TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 1, January 1985

FRACTIONAL INTEGRALS ON WEIGHTED H^p AND L^p SPACES

BY

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ABSTRACT. We study the two weight function problem $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$, $0 < p \leqslant q < \infty$, for fractional integrals on Hardy spaces. If u and v satisfy the doubling condition and $0 < p \leqslant 1$, we obtain a necessary and sufficient condition for the norm inequality to hold. If $1 < p < \infty$ we obtain a necessary condition and a sufficient condition, and show these are the same under various additional conditions on u and v. We also consider the corresponding problem for L_u^q and L_v^p, and obtain a necessary and sufficient condition in some cases.

1. Introduction. In this paper we study the behavior of fractional integrals on weighted Hardy spaces. In particular, we investigate the problem of determining pairs $u(x)$, $v(x)$ of nonnegative weight functions on $\mathbf{R}^n$ such that

$$\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}, \qquad 0 < p \leqslant q < \infty,$$

where I_α, $\alpha > 0$, is a fractional integral operator. We also give some applications of the results to the corresponding problem for L_u^q and L_v^p.

Following the usual terminology, we say that a weight function v satisfies the doubling condition if v is locally integrable and

$$v(2I) \leqslant cv(I), \tag{1.1}$$

where $v(I)$ is the v-measure of a cube I in $\mathbf{R}^n$ and $2I$ denotes the cube with the same center as I but with twice the edgelength. We write $v \in D_\infty$ for such v. Whenever dealing with Hardy spaces, we assume the weights satisfy the doubling condition. The definition of H_v^p, $0 < p < \infty$, is then the collection of tempered distributions f on $\mathbf{R}^n$ whose nontangential maximal function $(N_a f)(x)$ belongs to L_v^p, where L_v^p is the class of functions g with

$$\|g\|_{L_v^p} = \left(\int_{\mathbf{R}^n} |g(x)|^p v(x)\, dx\right)^{1/p} < \infty,$$

and

$$(N_a f)(x) = \sup_{(y,t):\, |x-y| < at} |(f * \phi_t)(y)|$$

with ϕ a Schwartz function, $\int_{\mathbf{R}^n} \phi(x)\, dx \neq 0$ and $\phi_t(x) = t^{-n}\phi(x/t)$, $t > 0$. Since $v \in D_\infty$, the definition of H_v^p is independent of the particular function ϕ and of the

Received by the editors June 29, 1983 and, in revised form, January 10, 1984.
1980 *Mathematics Subject Classification*. Primary 42B30, 42B25.

[1] Supported in part by NSF grant MCS80-03098.

0002-9947/85 $1.00 + $.25 per page

aperture a, and we may define $\|f\|_{H_v^p}$ by $\|f\|_{H_v^p} = \|f^*\|_{L_v^p}$, where f^* is the "grand" maximal function of f (see **[5** and **14]** for details). It is also true that $\|f\|_{H_v^p} \approx \|N_0 f\|_{L_v^p}$, where $(N_0 f)(x) = \sup_{t>0}|(f * \phi_t)(x)|$ is the radial maximal function.

A function $a(x)$ is called an (∞, N) atom if $|a(x)| \leqslant 1$, the support of $a(x)$ satisfies $\operatorname{supp} a \subset I$ where I is a cube, and $a(x)$ satisfies the moment conditions

$$\int_{\mathbf{R}^n} a(x) x^\gamma \, dx = 0, \qquad |\gamma| \leqslant N,$$

where N is a positive integer, $\gamma = (\gamma_1, \ldots, \gamma_n)$ with each γ_i a nonnegative integer, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. We shall often be interested in functions f which are multiples of (∞, N) atoms since these are dense in H_v^p (see Lemmas (2.1) and (2.2) for exact statements). For $\alpha > 0$ and such f, let

$$(I_\alpha f)(x) = \begin{cases} \displaystyle\int_{\mathbf{R}^n} f(x-y)|y|^{\alpha-n} \, dy, & \alpha \neq n + 2l, \\ \displaystyle\int_{\mathbf{R}^n} f(x-y) y |y|^{\alpha-n-1} \, dy, & \alpha = n + 2l, \end{cases}$$

$l = 0, 1, \ldots$. Note that when $\alpha = n + 2l$, $I_\alpha f$ is a vector. In case $\alpha = n + 2l$, the first integral above is zero if f is a multiple of an (∞, N) atom and N is large, and for that reason we modify the definition in this case. An alternate and different way to define $I_\alpha f$ in case $\alpha = n + 2l$ is to let $(I_\alpha f)(x) = f * |x|^{\alpha-n}\log|x|$.

It is easy to see (§2) that if f is any bounded function with compact support, then $I_\alpha f$ is bounded if $\alpha \leqslant n$ and has at most polynomial growth if $\alpha > n$. Moreover, if f is a multiple of an (∞, N) atom and N is sufficiently large, then the Fourier transform $[I_\alpha f]^\wedge$ of $I_\alpha f$ satisfies $[I_\alpha f]^\wedge(x) = c_{\alpha,n}|x|^{-\alpha}\hat{f}(x)$ if $\alpha \neq n + 2l$, $l = 0, 1, \ldots$ (see §2 for details).

For $\alpha > 0$ and $0 < p \leqslant q < \infty$, we consider the following condition on a pair (u, v) of weight functions:

$$(1.2) \qquad |I|^{\alpha/n} u(I)^{1/q} \leqslant c v(I)^{1/p},$$

where I is any cube in $\mathbf{R}^n$. We will also consider the condition

$$(1.3) \qquad \left\| \sum \lambda_k |I_k|^{\alpha/n} \chi_{I_k} \right\|_{L_u^q} \leqslant c \left\| \sum \lambda_k \chi_{I_k} \right\|_{L_v^p}, \qquad \lambda_k > 0,$$

where $\{I_k\}$ is any collection of cubes in $\mathbf{R}^n$ and χ_{I_k} denotes the characteristic function of I_k. It is easy to see that (1.3) implies (1.2) by considering the special case when the sum consists of just a single term. Also, as will be shown later, (1.2) implies $1/p - 1/q \leqslant \alpha/n$ unless $u = 0$ a.e.

Our main results for Hardy spaces are as follows:

THEOREM (1.4). *Let* $1 < p < \infty$, $\alpha > 0$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$ *and* $q < \infty$. *If* u, $v \in D_\infty$, *then* (1.2) *is necessary and* (1.3) *is sufficient for* $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$, *where* f *is any* (∞, N) *atom.*

THEOREM (1.5). *Let* $0 < p \leqslant 1$, $\alpha > 0$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$ *and* $q < \infty$. *If* u, $v \in D_\infty$, *then* (1.2) *is necessary and sufficient for* $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$, *where* f *is any* (∞, N) *atom.*

The sufficiency statements in these theorems are proved in §3 and the necessity statements in §4.

We note that if the inequality $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$ holds for all (∞, N) atoms, then it clearly also holds for multiples of (∞, N) atoms. For example, any finite linear combination of (∞, N) atoms is a multiple of an (∞, N) atom. Thus the norm inequality holds for all finite linear combinations of (∞, N) atoms, and, by Lemmas (2.1) and (2.2), these are dense in H_v^p. It follows that I_α has a continuous extension to all of H_v^p (see also the comments after Theorem (1.12)).

We shall not be precise about how large N must be chosen in the sufficiency statements. The value could be computed by keeping track of its size at different stages of the proof; it depends, for example, on the doubling order (see §2) of v and could be made smaller by further restricting v. The class of f's is dense in H_v^p no matter how large N is by Lemmas (2.1) and (2.2). For the necessity, N need only be taken large enough so that an (∞, N) atom is in H_v^p. Moreover, for the necessity in case $\alpha = n + 2l$, $l = 0, 1, \ldots,$ it is enough to assume the norm inequality for any component of I_α. One can also prove analogues of Theorems (1.4) and (1.5) in case $\alpha = n + 2l$ for the operator $f * |x|^{\alpha - n}\log|x|$. Such results can be stated as above without modifying the conditions.

In view of Theorem (1.4), it is of interest to know when (1.2) and (1.3) are equivalent. The next two theorems give relations between these conditions and the condition

$$(1.6) \quad |I|^{\alpha/n} u(I)^{1/p' + 1/q} \leqslant c \int_I u(x)^{1/p'} v(x)^{1/p}\, dx, \qquad 1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1.$$

We point out that (1.6) requires $1 < p < \infty$. We say that u and v are comparable if there exist $r, s, c_1, c_2 > 0$ such that

$$c_1\left[\frac{v(E)}{v(I)}\right]^r \leqslant \frac{u(E)}{u(I)} \leqslant c_2\left[\frac{v(E)}{v(I)}\right]^s$$

for any cube I and any measurable subset E of I. With the terminology of [3] this amounts to the statement that $v/u \in A_\infty(u\,dx)$ (or that $u/v \in A_\infty(v\,dx)$). Either of the two inequalities above, of course, implies the other. We refer the reader to [**3** or **11**] for the definition and properties of the spaces $A_\infty(d\mu)$ and $A_p(d\mu)$.

THEOREM (1.7). (i) (1.6) $\Rightarrow$ (1.2) *if* $1 < p < \infty$, *and* (1.6) $\Rightarrow$ (1.3) *if* $1 < p < \infty$ *and* $u \in D_\infty$.

(ii) (1.2) $\Rightarrow$ (1.6) *if u and v are comparable and* $1 < p < \infty$. *Thus*, (1.2) $\Rightarrow$ (1.6) *if* $u = v^{q/p}$, $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$.

(iii) *Any of the three conditions implies that* $1/p - 1/q \leqslant \alpha/n$ *unless* $u = 0$ *a.e.*

In particular, (1.6) $\Rightarrow$ (1.3) $\Rightarrow$ (1.2) *if* $1 < p < \infty$ *and* $u \in D_\infty$, *and all three conditions are equivalent if u and v are comparable and* $u \in D_\infty$. *For example, all three are equivalent if* $u = v^{q/p}$, $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$, $u \in D_\infty$.

Theorem (1.7) is proved in §5.

In case $q > p$ we also have the following results relating (1.2) and (1.3). A weight v is said to satisfy the reverse Hölder condition of order $r > 1$, that is, $v \in \mathrm{RH}_r$, if

$$\left(\frac{1}{|I|}\int_I v(x)^r\,dx\right)^{1/r} \leqslant c\frac{1}{|I|}\int_I v(x)\,dx.$$

This condition is easily seen to be equivalent to $v^r \in A_\infty$ (see Corollary (6.2); by A_∞ we mean $A_\infty(dx)$).

THEOREM (1.8). *If* $1 < p < q$ *then* (1.2) *and* (1.3) *are equivalent if either* $v \in \mathrm{RH}_{p^*/p}$, $1/p^* = 1/p - \alpha/n, p < n/\alpha$, *or if* $u \in A_\infty$ *and* $v \in D_\infty$.

Thus, for example, (1.2) and (1.3) are equivalent for $1 < p < n/\alpha$ and $q > p$ if $v \equiv 1$ or if $u \equiv 1$ and $v \in D_\infty$. Theorem (1.8) is proved in §5.

We now give an example of a pair of weights satisfying the conditions.

THEOREM (1.9). *Let* $1 < p < \infty$, $\alpha > 0$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$ *and* $\beta/n = \alpha/n - (1/p - 1/q)$. *Let* $\{a_k\}_{k=1}^m$ *be distinct points of* $\mathbf{R}^n$ *and, for* $\mu_k > 0$ *and* $M \geqslant \beta$, *define*

$$\Pi(x) = (1+|x|)^M\prod_{k=1}^m\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\mu_k},$$

$$\Pi_\beta(x) = (1+|x|)^{M-\beta}\prod_{k=1}^m\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\gamma_k}, \qquad \gamma_k = \max\{\mu_k - \beta, 0\}.$$

If $w \in D^\infty$ and, when $q > p$, if $w \in \mathrm{RH}_{q/p}$, then the pair $(u, v) = (\Pi_\beta^q w^{q/p}, \Pi^p w)$ satisfies (1.6) and $u, v \in D_\infty$.

For example, if the dimension $n = 1$, and we choose each μ_k to be a positive integer and $M = \Sigma\mu_k$, then $\Pi(x)$ is equivalent in size to $|Q(x)|$, where $Q(x)$ is the polynomial $Q(x) = \Pi(x - a_k)^{\mu_k}$. In this case, $\Pi_\beta(x)$ is equivalent to

$$|Q(x)|(1+|x|)^{-\beta}\prod\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{-\beta_k}$$

where $\beta_k = \min\{\mu_k, \beta\}$.

Theorem (1.9) is proved in §6, as is the following corollary of its proof.

THEOREM (1.10). *Let* $0 < p < \infty$, $\alpha > 0$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$ *and* $\beta/n = \alpha/n - (1/p - 1/q)$. *Let* $\{a_k\}_{k=1}^m$ *be distinct points of* $\mathbf{R}^n$ *and, for* $0 \leqslant \nu_k \leqslant \beta$, *define*

$$\Pi^*(x) = (1+|x|)^\beta\prod_{k=1}^m\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\nu_k}.$$

Then the pair $(u, v) = ([v^{1/p}/\Pi^*]^q, v)$ *satisfies* $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$ *if* $u, v \in D_\infty$ *and, in addition, when* $q \neq p$, *if* $u \in A_\infty$. *In fact, under these conditions* (u, v) *satisfies* (1.6) *if* $1 < p < \infty$ *and* (1.2) *if* $0 < p \leqslant 1$.

Although we have so far only considered the behavior of fractional integrals on weighted H^p spaces, our theorems also yield results for weighted L^p spaces, $1 < p < \infty$, if we restrict the weight function v which appears on the right to be one for which H_v^p and L_v^p coincide. Such weights have been studied in **[16]** for $n = 1$ and in **[2]** for $n > 1$. For example, Theorem (1.4) remains true for such v if H_v^p is replaced by L_v^p and H_u^q is replaced by L_u^q. Note that $\|I_\alpha f\|_{L_u^q} \leqslant c\|I_\alpha f\|_{H_u^q}$ if f is an atom, since then $I_\alpha f \leqslant (I_\alpha f)^*$ a.e. due to the fact that $I_\alpha f$ is locally integrable. Actually, for the sufficiency statement, if we deal directly with L_u^q, the hypothesis that $u \in D_\infty$ can be weakened: see the argument in §7. In §7 we will prove the following result concerning *general* functions f in L_v^p when $v \in A_p$.

THEOREM (1.11). *Let* $1 < p < \infty, 0 < \alpha < n$ *and* $0 \leqslant 1/p - 1/q \leqslant \alpha/n$. *If* $v \in A_p$ *and* $I_\alpha f = f * |x|^{\alpha-n}$, *then condition* (1.3) *is necessary and sufficient for the inequality* $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$.

In this result it is not assumed that $u \in D_\infty$.

By combining Theorems (1.11) and (1.8) we obtain conditions under which (1.2) is necessary and sufficient for $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ for general f. Thus, if $1 < p < q$ and $v \in A_p$, we see (1.2) is necessary and sufficient if either $v \in A_{p^*/p}$, $1/p^* = 1/p - \alpha/n$, $p < n/\alpha$, or if $u \in A_\infty$. For example, if $1 < p < n/\alpha$, $q > p$ and $v \equiv 1$, then (1.2) is necessary and sufficient for $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L^p}$. This is the main result of **[1]**. The main result of **[12]** also follows in this way, as will be shown later. Other necessary and sufficient conditions are given in **[4, 7 and 8]**, but it is not obvious how these are related to (1.3).

As a specific example of the kind of results which hold when $v \notin A_p$, we mention the following theorem for the case $n = 1$.

THEOREM (1.12). *Let* $1 < p < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha$ *and* $\beta = \alpha - (1/p - 1/q)$. *Let* $Q(x) = \prod_{k=1}^m (x - a_k)^{\mu_k}$ *be a polynomial with distinct real roots* $\{a_k\}$ *and order* $M \geqslant \beta$. *Let* $v(x) = |Q(x)|^p w(x)$ *and*

$$u(x) = \left\{|Q(x)|(1 + |x|)^{-\beta} \prod_{k=1}^m \left(\frac{|x - a_k|}{1 + |x - a_k|}\right)^{-\beta_k}\right\}^q w(x)^{q/p}$$

where $\beta_k = \min(\mu_k, \beta)$, $w \in A_p$, *and, in addition, when* $q \neq p$, $w \in \mathrm{RH}_{q/p}$. *Let* $R_\alpha^+ f$ *be defined by*

$$R_\alpha^+ f(x) = \int_{-\infty}^x f(y)(x - y)^{\alpha-1}\,dy$$

when f *is an* (∞, N) *atom. Then* $\|R_\alpha^+ f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ *for such* f.

A similar result holds for the transform

$$R_\alpha^- f(x) = \int_x^\infty f(y)(y - x)^{\alpha-1}\,dy,$$

and there are n-dimensional versions. We shall not investigate such results in this paper but postpone them until **[17]**, where a direct proof not depending on the main results of **[16]** will be given. This direct proof is based on kernel estimates and also shows how $R_\alpha^+ f$ is to be defined for general f in L_v^p. It is necessary to modify the definition of $R_\alpha^+ f$ for general f since there may be functions in L_v^p which are not even locally integrable.

We would like to point out that special cases of our results imply the main theorems of Macías-Segovia [**10**] and Muckenhoupt-Wheeden [**12**]. These theorems are concerned with the case when $u = v^{q/p}$ and $1/q = 1/p - \alpha/n$ (so $\beta = 0$). See the short remark at the end of the paper for details.

In passing, we would like to thank Dr. C. E. Gutierrez and Dr. A. E. Gatto for several helpful comments about the proof of Lemma (6.6).

2. Background facts. In this section we list several lemmas about weight functions and H^p spaces which will be useful in proving the sufficiency parts of Theorems (1.4) and (1.5). Some other facts about weight functions needed for Theorem (1.9) are given in §6.

We first note, as mentioned in the Introduction, that if f is any bounded function with compact support, then $I_\alpha f$ is bounded if $\alpha \leqslant n$ and has at most polynomial growth if $\alpha > n$. In fact, if f is supported in $|y| < R$, then

$$|(I_\alpha f)(x)| \leqslant \|f\|_{L^\infty} \int_{|y|<R} |x-y|^{\alpha-n}\, dy \leqslant c_R \|f\|_{L^\infty}(1+|x|)^{\alpha-n}.$$

If f also satisfies moment conditions, we will see later than $I_\alpha f$ decays more rapidly at ∞. We also note that if f is a multiple of an (∞, N) atom and N is sufficiently large, then the Fourier transform $[I_\alpha f]\hat{}$ of $I_\alpha f$ satisfies

$$[I_\alpha f]\hat{}(x) = \begin{cases} c_{\alpha,n}|x|^{-\alpha}\hat{f}(x) & \text{if } \alpha \neq n+2l,\ l = 0, 1, \ldots, \\ c_{\alpha,n}x|x|^{-\alpha-1}\hat{f}(x) & \text{if } \alpha = n+2l. \end{cases}$$

Here, the Fourier transform is taken componentwise in case $\alpha = n + 2l$. Of course, no moment conditions are required when $0 < \alpha < n$. These formulas may be derived by using [**6**, Volume 1, pp. 194–195]. See also [**14**, p. 117] when $0 < \alpha < n$. It is shown in [**6**] that the function whose Fourier transform is $|x|^{-\alpha}\hat{f}(x)$ for $\alpha = n + 2l$ is a constant times $f * |x|^{\alpha-n}\log|x|$.

We now list three lemmas used in the next section. A weight $v \in D_\infty$ is said to belong to D_s, $1 \leqslant s < \infty$, if $v(tI) \leqslant ct^{ns}v(I)$ for all $t > 1$, where tI is the cube concentric with I whose edgelength is t times the edgelength of I. Clearly, $D_\infty = \bigcup_{s \geqslant 1} D_s$. We will also use in §6 the simple fact that any $v \in D_\infty$ satisfies a reverse doubling condition, denoted by $v \in \mathrm{RD}_\varepsilon$, $\varepsilon > 0$, which means that $v(tI) \geqslant ct^{n\varepsilon}v(I)$ for all $t > 1$. Finally, for a real number r, int r denotes the greatest integer $\leqslant r$.

LEMMA (2.1). (i) *If $v \in D_\infty$, $1 \leqslant p < \infty$, $N > 0$ and $f \in H^p_v$, then $f = \sum \lambda_k a_k$, where $\lambda_k > 0$, a_k is an (∞, N) atom supported in a cube I_k, the sum converges in H^p_v norm and in the sense of distributions, and $\|\sum \lambda_k \chi_{I_k}\|_{L^p_v} \leqslant c\|f\|_{H^p_v}$.*

(ii) *Let $v \in D_s$, $1 \leqslant p < \infty$, $\tilde{N} = \operatorname{int} s(n-1)$ and $\{a_k\}$ be (∞, N) atoms with $N \geqslant \tilde{N}$ and a_k supported in I_k. Then if $\sum \lambda_k \chi_{I_k} \in L^p_v$, $\sum \lambda_k a_k$ converges in H^p_v and $\|\sum \lambda_k a_k\|_{H^p_v} \leqslant c\|\sum \lambda_k \chi_{I_k}\|_{L^p_v}$.*

This lemma as well as the next one are proved in [**15**]. The condition on the size of N can be relaxed if w also belongs to A_r (see [**15**]).

LEMMA (2.2). *Let $v \in D_s$, $0 < p \leqslant 1$, $\tilde{N} = \operatorname{int} s(n-1)/p$. If $f \in H_v^p$, then $f = \sum \lambda_k a_k$, where $\lambda_k > 0$, a_k is an (∞, N) atom with support in I_k, $N \geqslant \tilde{N}$, the sum converges in H_v^p and in the sense of distributions, and $(\sum \lambda_k^p v(I_k))^{1/p} \approx \|f\|_{H_v^p}$. The converse also holds.*

Finally, we have

LEMMA (2.3). *If $v \in D_s$, then for $1 \leqslant p < \infty$, $\lambda_k > 0$ and $t > 1$,*

$$\Big\| \sum \lambda_k \chi_{tI_k} \Big\|_{L_v^p} \leqslant c t^{ns} \Big\| \sum \lambda_k \chi_{I_k} \Big\|_{L_v^p}.$$

This lemma, which is also in [15], can be proved as follows. If $p = 1$, the left side equals

$$\sum \lambda_k v(tI_k) \leqslant c t^{ns} \sum \lambda_k v(I_k) = c t^{ns} \Big\| \sum \lambda_k \chi_{I_k} \Big\|_{L_v^1}.$$

For $1 < p < \infty$,

$$\Big\| \sum \lambda_k \chi_{tI_k} \Big\|_{L_v^p} = \sup \int \sum \lambda_k \chi_{tI_k} g v \, dx,$$

the sup being taken over all $g \geqslant 0$ with $\|g\|_{L_v^{p'}} = 1$. But,

$$\int \sum \lambda_k \chi_{tI_k} g v \, dx = \sum \lambda_k \left[\int_{tI_k} \frac{g v \, dx}{v(tI_k)} \right] v(tI_k)$$

$$\leqslant c t^{ns} \sum \lambda_k \left[\int_{tI_k} \frac{g v \, dx}{v(tI_k)} \right] v(I_k) \leqslant c t^{ns} \int \Big(\sum \lambda_k \chi_{I_k} \Big) M_v(g) v \, dx,$$

where $M_v(g)$ is the Hardy-Littlewood maximal function of g with respect to v-measure:

$$M_v(g)(x) = \sup_{I:\, I \ni x} \frac{1}{v(I)} \int_I g v \, dx.$$

By Hölder's inequality, the last expression is at most

$$c t^{ns} \Big\| \sum \lambda_k \chi_{I_k} \Big\|_{L_v^p} \|M_v(g)\|_{L_v^{p'}},$$

and the lemma follows since $\|M_v(g)\|_{L_v^{p'}} \leqslant c \|g\|_{L_v^{p'}} = c$.

3. Sufficiency for Theorems (1.4) and (1.5). We begin by proving the sufficiency part of Theorem (1.4). Thus, let $\alpha > 0$, $1 < p$, $q < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$, $u \in D_s$ and $v \in D_\infty$. Let a be an (∞, N) atom supported in a cube $I \subset \{x \colon |x| < r\}$ with $|I| \sim r^n$. Write

$$(I_\alpha a)(x) = \int_{\mathbf{R}^n} a(y) K(x-y) \, dy,$$

where $K(x) = |x|^{\alpha - n}$ if $\alpha \neq n + 2l$, $l = 0, 1, 2, \ldots$, and $K(x) = (x_j/|x|)|x|^{\alpha-n}$, $j = 1, \ldots, n$, if $\alpha = n + 2l$. An argument for the alternate definition in case $\alpha = n + 2l$ is given at the end of this section. Decompose $K(x) = \sum_{i=0}^\infty K_i(x)$, where K_0 is supported in $|x| < 2r$, K_i in $2^{i-1} r < |x| < 2^{i+1} r$, $i = 1, 2, \ldots$, and $|(\partial/\partial x)^\beta K_i(x)| \leqslant c|x|^{\alpha - n - |\beta|}$ for any multi-index β. Then

$$(I_\alpha a)(x) = \sum_{i=0}^\infty \int a(y) K_i(x-y) \, dy \equiv \sum_{i=0}^\infty b_i(x).$$

Note that b_0 is supported in $|x| < 3r$, b_i is supported in a ring comparable to that supporting K_i for $i \geqslant 1$, and b_i satisfies the same moment conditions as a:

$$\int b_i(x)x^\beta\,dx = \int\left(\int a(x-y)x^\beta\,dx\right)K_i(y)\,dy$$
$$= \int\left(\int a(x)(x+y)^\beta\,dx\right)K_i(y)\,dy = 0$$

for $|\beta| \leqslant N$, since $(x+y)^\beta$ is a polynomial in x of degree $|\beta|$.

We will now estimate $\|b_i\|_\infty$. For $i = 0, 1$ or 2, we have

$$|b_i(x)| \leqslant \int |a(y)|\,|K_i(x-y)|dy \leqslant c\int_{|x-y|<2^3r} |x-y|^{\alpha-n}\,dy = cr^\alpha.$$

For $i \geqslant 3$ use Taylor's theorem to write

$$K_i(x-y) = \sum_{|\beta|\leqslant N} \frac{1}{\beta!}\left[\left(\frac{\partial}{\partial x}\right)^\beta K_i(x)\right](-y)^\beta + R(x,y).$$

If $|y| < r$ and $2^{i-1}r < |x-y| < 2^{i+1}r$, then $|x| \sim 2^i r$ and, in particular, $|x| > 2|y|$. Thus, for such x and y,

$$|R(x,y)| \leqslant c_N|y|^{N+1}|x|^{\alpha-n-N-1} \leqslant cr^{N+1}(2^ir)^{\alpha-n-N-1} = cr^{\alpha-n}2^{i(\alpha-n-N-1)}.$$

Since $\int a(y)y^\beta\,dy = 0$ for $|\beta| \leqslant N$, we obtain

$$|b_i(x)| = \left|\int a(y)R(x,y)\,dy\right| \leqslant cr^{\alpha-n}2^{i(\alpha-n-N-1)}\int |a(y)|dy$$
$$\leqslant cr^\alpha 2^{i(\alpha-n-N-1)} \equiv \mu_i.$$

Collecting facts we see that

$$(I_\alpha a)(x) = \sum_0^\infty b_i(x) = \sum_0^\infty \mu_i\left(\frac{b_i(x)}{\mu_i}\right), \tag{3.1}$$

where $b_i(x)/\mu_i$ is an (∞, N) atom supported in $c2^iI$. Moreover, for any x, $b_i(x) = 0$ except for at most 3 values of i. We also note that $\sum b_i$ converges as a distribution to $I_\alpha a$: in fact, if ϕ is a Schwartz function, then

$$\sum_{i>J}\int |b_i\phi|dx \leqslant \|\phi\|_{L^1}\sum_{i>J}\mu_i \to 0$$

as $J \to \infty$ if $N > \alpha - n - 1$.

Now let f be any (∞, N) atom. By Lemma (2.1) we can write $f = \sum_1^\infty \lambda_k a_k$ with $\lambda_k > 0$, a_k an (∞, N) atom supported in I_k and $\|\sum_1^\infty \lambda_k\chi_{I_k}\|_{L^p_v} \leqslant c\|f\|_{H^p_v}$. Let $f_M = \sum_1^M \lambda_k a_k$. By what was shown above, we have

$$I_\alpha a_k = \sum_i b_{k,i} = \sum_i \mu_{k,i}\left(\frac{b_{k,i}}{\mu_{k,i}}\right),$$

where $\mu_{k,i} = c|I_k|^{\alpha/n}2^{i(\alpha-n-N-1)}$ and $b_{k,i}/\mu_{k,i}$ is an (∞, N) atom supported in $c2^iI_k$.

Thus,

$$I_\alpha f_M = \sum_{k=1}^{M} \lambda_k (I_\alpha a_k) = \sum_{k=1}^{M} \lambda_k \sum_i \mu_{k,i} \left(\frac{b_{k,i}}{\mu_{k,i}} \right).$$

We note that the summation in k is extended only over a finite number of k's, and that for each x and k, there are at most 3 values of i for which $b_{k,i}(x) \neq 0$. Hence, the double sum converges pointwise to $I_\alpha f_M$. Moreover, it converges to $I_\alpha f_M$ as a distribution since each $\sum_i b_{k,i}$ converges as a distribution to $I_\alpha a_k$.

Note that

$$\left\| \sum_{k,i} \lambda_k \mu_{k,i} \chi_{c2^i I_k} \right\|_{L^q_u} \leqslant \sum_i \left\| \sum_k \lambda_k \mu_{k,i} \chi_{c2^i I_k} \right\|_{L^q_u}$$
$$\leqslant c \sum_i \left\| \sum_k \lambda_k \mu_{k,i} \left| c2^i I_k \right|^{-\alpha/n} \chi_{c2^i I_k} \right\|_{L^p_v}$$

by condition (1.3) applied to $\{c2^i I_k\}_k$. Since $v \in D_t$ for some t, Lemma (2.3) shows that the last expression is bounded by

$$c \sum_i 2^{int} \left\| \sum_k \lambda_k \mu_{k,i} \left| c2^i I_k \right|^{-\alpha/n} \chi_{I_k} \right\|_{L^p_v}.$$

Substituting the value of $\mu_{k,i}$ and noting that $|2^i I_k|^{-\alpha/n} = 2^{-i\alpha} |I_k|^{-\alpha/n}$, we get at most

$$c \sum_i 2^{int} 2^{-i\alpha} 2^{i(\alpha - n - N - 1)} \left\| \sum_k \lambda_k \chi_{I_k} \right\|_{L^p_v} \leqslant c \|f\|_{H^p_v},$$

provided $N > nt - n - 1$. By Lemma (2.1) (ii) applied to $I_\alpha f_M$ with $N \geqslant \text{int}\, s(n-1)$, $u \in D_s$, it follows that

$$\|I_\alpha f_M\|_{H^q_u} \leqslant c \|f\|_{H^p_v} \tag{3.2}$$

with c independent of M and f.

To complete the proof that $\|I_\alpha f\|_{H^q_u} \leqslant c \|f\|_{H^p_v}$ for such f, we claim that the atomic decomposition of f can be chosen so that $I_\alpha f_M \to I_\alpha f$ in H^q_u as $M \to \infty$. This will clearly suffice. The argument needed to prove this claim is mainly dominated convergence combined with the special nature of f. Let g^* denote the grand maximal function of g. By **[15]** the atomic decomposition may be chosen so that the partial sums $f_M = \sum_1^M \lambda_k a_k$ converge in (Lebesgue) measure to f, $\lambda_k > 0$ and $\sum_0^\infty \lambda_k \chi_{I_k} \leqslant cf^*$. This will be enough to show the claim if we also use the following majorizations:

$$|f(x)| \leqslant c(1+|x|)^{-L}, \tag{3.3}$$

$$f^*(x) \leqslant c(1+|x|)^{-L}, \tag{3.4}$$

$$|I_\alpha f(x)| \leqslant c(1+|x|)^{-L}, \tag{3.5}$$

$$(I_\alpha f)^*(x) \leqslant c(1+|x|)^{-L}, \tag{3.6}$$

$$|f_M(x)| \leqslant cf^*(x) \leqslant c(1+|x|)^{-L}, \tag{3.7}$$

$$f_M^*(x) \leqslant c(1+|x|)^{-L}, \tag{3.8}$$

$$|(I_\alpha f_M)(x)| \leqslant c(1+|x|)^{-L}, \tag{3.9}$$

$$(I_\alpha f_M)^*(x) \leqslant c(1+|x|)^{-L}. \tag{3.10}$$

Here L may be chosen as large as desired by taking N large, and the last four estimates are uniform in M. Inequality (3.3) is obvious for any L since f is bounded and has compact support. The next four inequalities can easily be derived from (3.3) and the fact that a large number of moments of f vanish. Let us show (3.5) for $\alpha \neq n + 2l$ for example. First note that if $|x| \leqslant 1$, then

$$\begin{aligned} |I_\alpha f(x)| &\leqslant c\left(\int_{|y|>2} (1+|y|)^{-L'}|y|^{\alpha-n}\,dy + \int_{|y|<2} |x-y|^{\alpha-n}\,dy\right) \\ &\leqslant c\left(\int_{\mathbf{R}^n} (1+|y|)^{-L'}|y|^{\alpha-n}\,dy + \int_{|y|<3} |y|^{\alpha-n}\,dy\right) < \infty \end{aligned}$$

since $\alpha > 0$. This shows that $I_\alpha f(x)$ is bounded for $|x| \leqslant 1$. If $|x| > 1$ we use the moment conditions to write

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)\left\{|x-y|^{\alpha-n} - \sum_{|\beta|\leqslant N} \frac{1}{\beta!}\left[\left(\frac{\partial}{\partial x}\right)^\beta |x|^{\alpha-n}\right](-y)^\beta\right\}dy.$$

We will consider the three domains of integration, $|y| < |x|/2$, $|y| > 2|x|$ and $|x|/2 \leqslant |y| \leqslant 2|x|$, separately. If $|y| < |x|/2$, Taylor's theorem gives the bound

$$\frac{c}{|x|^{N+1+n-\alpha}}\int |f(y)|\,|y|^{N+1}\,dy.$$

Given L, pick N so large that $N + 1 + \mathrm{n} - \alpha \geqslant L$. Then using the estimate $|f(y)| \leqslant c(1+|y|)^{-L'}$, with $L' > N + 1 + n$, shows that $\int |f(y)|\,|y|^{N+1}dy < \infty$, and the desired estimate follows in this case. If $|y| > 2|x|$, then both $|x|, |y| \geqslant 1$, and the integral is bounded by

$$\begin{aligned} &c\int_{|y|>2|x|} |f(y)|\left[|y|^{\alpha-n} + |x|^{\alpha-n}|y|^N\right]dy \\ &\qquad \leqslant c\int_{|y|>2|x|} (1+|y|)^{-L'}\left[|y|^{\alpha-n} + |x|^{\alpha-n}|y|^N\right]dy. \end{aligned}$$

The desired estimate now follows easily by choosing L' sufficiently large and using

$$(1+|y|)^{-L'} \leqslant (1+|x|)^{L}(1+|y|)^{L'-L}.$$

Finally, the part of the integral with $|x|/2 \leqslant |y| \leqslant 2|x|$ is bounded by

$$\begin{aligned} &c\int_{|x|/2\leqslant|y|\leqslant 2|x|} |y|^{-L'}\left[|x-y|^{\alpha-n} + |x|^{\alpha-n}\right]dy \\ &\qquad \leqslant c|x|^{-L'}\int_{|y|\leqslant 2|x|} \left[|x-y|^{\alpha-n} + |x|^{\alpha-n}\right]dy \leqslant c|x|^{-L'+\alpha}. \end{aligned}$$

Combining estimates, we obtain (3.5).

To derive inequalities (3.8)–(3.10) we fix M and group the atoms as follows. Let $\tilde{I}_0$ be the unit cube centered at the origin, and let $\tilde{I}_j = 2^j I_0$, $j = 1, 2, \ldots$. Define

$$\tilde{\nu}_0\tilde{a}_0 = \sum_{\substack{k=1 \\ I_k \subset \tilde{I}_0}}^{M} \lambda_k a_k \quad \text{and} \quad \tilde{\nu}_j\tilde{a}_j = \sum_{\substack{k=1 \\ I_k \subset \tilde{I}_j,\, I_k \not\subset \tilde{I}_{j-1}}}^{M} \lambda_k a_k, \qquad j = 1, 2, \ldots,$$

where $\tilde{\nu}_j$ is the L^∞ norm of

$$\sum_{\substack{k=1 \\ I_k \subset \tilde{I}_j,\, I_k \not\subset \tilde{I}_{j-1}}}^{M} \lambda_k \chi_{I_k}(x).$$

Thus, $\tilde{a}_j$ is an (∞, N) atom supported in $\tilde{I}_j$, and $\sum_{k=1}^{M} \lambda_k a_k = \sum_{j=0}^{M} \tilde{\nu}_j \tilde{a}_j$. To estimate $\tilde{\nu}_j$, note first that if $|x| \geqslant 2^{j-4}$, then

$$\sum_{k=1}^{M} \lambda_k \chi_{I_k}(x) \leqslant cf^*(x) \leqslant c(1+|x|)^{-L'} \leqslant c2^{-jL'}.$$

If $|x| \leqslant 2^{j-3}$ the edgelength of any I_k such that $\chi_{I_k}(x) \neq 0$ and $I_k \not\subset \tilde{I}_{j-1}$ must be at least 2^{j-3}, so I_k must contain at least one of the 2^n points $y_l = (2^{j-4}\varepsilon_1, \ldots, 2^{j-4}\varepsilon_n)$, where each ε_i is either 1 or -1. Thus, if $|x| \leqslant 2^{j-3}$,

$$\sum_{\substack{k=1 \\ I_k \subset \tilde{I}_j,\, I_k \not\subset \tilde{I}_{j-1}}}^{M} \lambda_k \chi_{I_k}(x) \leqslant \sum_{l=1}^{2^n} \sum_{k=1}^{M} \lambda_k \chi_{I_k}(y_l) \leqslant \sum_{l=1}^{2^n} c2^{-jL'} = c2^{-jL'}$$

by the above estimate since $|y_l| \geqslant 2^{j-4}$. It follows that $\tilde{\nu}_j \leqslant c2^{-jL'}$ with c independent of j.

Since $\tilde{a}_j^*(x) \leqslant c(1+|x|2^{-j})^{-L'}$ with c independent of j, we have

$$\left(\sum_{k=1}^{M} \lambda_k a_k\right)^*(x) = \left(\sum_{j=0}^{M'} \tilde{\nu}_j \tilde{a}_j\right)^*(x) \leqslant c\sum_{j=0}^{\infty} 2^{-jL'}(1+|x|2^{-j})^{-L'} \leqslant c(1+|x|)^{-L}$$

for any $L < L'$, which gives (3.8). Similarly,

$$|I_\alpha \tilde{a}_j(x)| \leqslant c2^{j\alpha}(1+|x|2^{-j})^{-L'},$$

and we get (3.9) by summation. Finally, writing $I_\alpha \tilde{a}_j$ as a sum of atoms as in (3.1), we have

$$(I_\alpha \tilde{a}_j)(x) = 2^{j\alpha} \sum_{i=0}^{\infty} 2^{-i(N+1+n-\alpha)} \tilde{b}_{j,i}(x),$$

where $\tilde{b}_{j,i}$ is an atom supported in $c2^i \tilde{I}_j$ and

$$\begin{aligned}(I_\alpha \tilde{a}_j)^*(x) &\leqslant 2^{j\alpha} \sum_{i=0}^{\infty} 2^{-i(N+1+n-\alpha)} (\tilde{b}_{j,i})^*(x) \\ &\leqslant c2^{j\alpha} \sum_{i=0}^{\infty} 2^{-i(N+1+n-\alpha)} (1+|x|2^{-i-j})^{-L'} \leqslant c2^{j\alpha}(1+|x|2^{-j})^{-L'}\end{aligned}$$

by choosing N sufficiently large depending on L'. Summation over j now gives (3.10).

To see how (3.3)–(3.10) imply that $I_\alpha f_M$ converges to $I_\alpha f$ in H_u^q, first note that $I_\alpha f_M$ converges pointwise to $I_\alpha f$ by using the facts that f_M converges in measure to f, (3.3), (3.7) and

$$\int_{\mathbf{R}^n} (1+|y|)^{-L} |x-y|^{\alpha-n}\, dy < \infty \qquad (L > \alpha).$$

Since $u \in D_s$, we have

$$\int_{\mathbf{R}^n} (1 + |x|)^{-Lq} u(x)\, dx < \infty \tag{3.11}$$

provided $L > ns/q$. Thus, (3.5) and (3.9) together with the Lebesgue dominated convergence theorem imply that $I_\alpha f_M$ converges to $I_\alpha f$ in L_u^q as $M \to \infty$. In particular, this convergence is in L^2, so that $\|[I_\alpha(f - f_M)]^*\|_{L^2} \to 0$. Note by (3.6) and (3.10) that $[I_\alpha(f - f_M)]^*(x) \leqslant c(1 + |x|)^{-L}$ with c independent of M. Given $\varepsilon > 0$, pick δ such that

$$\int_{|x| > 1/\delta} (1 + |x|)^{-Lq} u(x)\, dx < \varepsilon \quad \text{and} \quad \int_{\{x:\, u(x) > 1/\delta\}} (1 + |x|)^{-Lq} u(x)\, dx < \varepsilon,$$

noting for the second of these that the measure of the part of $\{x\colon u(x) > 1/\delta\}$ in any bounded set tends to zero with δ. Thus,

$$\begin{aligned}
\|[I_\alpha f - I_\alpha f_M]^*\|_{L_u^q}^q &= \int_{\mathbf{R}^n} [I_\alpha(f - f_M)]^*(x)^q u(x)\, dx \\
&\leqslant \int_{|x| > 1/\delta} + \int_{\{x:\, u(x) > 1/\delta\}} + \int_{\{x:\, |x| \leqslant 1/\delta,\ u(x) \leqslant 1/\delta\}} \\
&\leqslant 2c^q \varepsilon + \frac{1}{\delta} \int_{|x| < 1/\delta} [I_\alpha(f - f_M)]^*(x)^q\, dx.
\end{aligned}$$

If $r = \min(q, 2)$, the last term on the right is bounded by

$$\frac{1}{\delta} c^{q-r} \int_{|x| < 1/\delta} [I_\alpha(f - f_M)]^{*r}\, dx,$$

which by Hölder's inequality is at most

$$\delta^{-1} c^{q-r} \|[I_\alpha(f - f_M)]^*\|_{L^2}^r \delta^{-2n/(2-r)}.$$

This tends to zero as $M \to \infty$, and we conclude that

$$\|I_\alpha f - I_\alpha f_M\|_{H_u^q} = \|[I_\alpha f - I_\alpha f_M]^*\|_{L_u^q} \to 0$$

as $M \to \infty$. Thus, our earlier claim is established and the proof of the sufficiency part of Theorem (1.4) is complete.

The proof of the sufficiency part of Theorem (1.5) is similar. Now $0 < p \leqslant 1$ and we consider the cases $q \geqslant 1$ and $q < 1$ separately. In either case, for an (∞, N) atom f, we use Lemma (2.2) to write $f = \sum_1^\infty \lambda_k a_k$ with $\lambda_k > 0$, a_k an atom supported in I_k, and $(\sum_1^\infty \lambda_k^p v(I_k))^{1/p} \leqslant c\|f\|_{H_v^p}$. If $q \geqslant 1$ and $f_M = \sum_1^M \lambda_k a_k$, we argue as before to obtain

$$\|I_\alpha f_M\|_{H_u^q} \leqslant c \Big\| \sum_{k,i} \lambda_k \mu_{k,i} \chi_{c2^i I_k} \Big\|_{L_u^q}.$$

Using Minkowski's inequality, the definition of $\mu_{k,i}$ and the fact that $u \in D_s$, we see the right side is bounded by

$$c \sum_{k\ i} \lambda_k 2^{i(\alpha - n - N - 1)} |I_k|^{\alpha/n} u(c2^i I_k)^{1/q} \leqslant c \sum_{k,i} 2^{i(\alpha - n - N - 1)} 2^{ins/q} \lambda_k |I_k|^{\alpha/n} u(I_k)^{1/q}.$$

Choosing $N > \alpha - n - 1 + ns/q$ and performing the summation in i shows the last sum equals

$$c\sum_k \lambda_k |I_k|^{\alpha/n} u(I_k)^{1/q} \leqslant c\sum_k \lambda_k v(I_k)^{1/p}$$

by condition (1.2). Since $0 < p \leqslant 1$,

$$\sum_k \lambda_k v(I_k)^{1/p} \leqslant \left[\sum_k \lambda_k^p v(I_k)\right]^{1/p}$$

and we obtain

$$\|I_\alpha f_M\|_{H_u^q} \leqslant c\left[\sum_k \lambda_k^p v(I_k)\right]^{1/p}. \tag{3.12}$$

The right side of (3.12) is at most $c\|f\|_{H_v^p}$, and we have verified (3.2). The rest of the proof works as before.

If, on the other hand, $q < 1$, then the fact that

$$I_\alpha f_M = \sum_{k=1}^{M} \sum_i \lambda_k \mu_{k,i} \left(\frac{b_{k,i}}{\mu_{k,i}}\right),$$

where $b_{k,i}/\mu_{k,i}$ is an atom supported on $c2^i I_k$, implies that

$$\|I_\alpha f_M\|_{H_u^q} \leqslant c\left[\sum_{k,i} (\lambda_k \mu_{k,i})^q u\left(c2^i I_k\right)\right]^{1/q}.$$

Since $u \in D_s$, the right side is at most

$$c\left[\sum_{k,i} (\lambda_k \mu_{k,i})^q 2^{ins} u(I_k)\right]^{1/q} = c\left[\sum_k \left(\lambda_k |I_k|^{\alpha/n}\right)^q \left(\sum_i 2^{i(\alpha-n-N-1)q+ins}\right) u(I_k)\right]^{1/q}$$

$$\leqslant c\left[\sum_k \left(\lambda_k |I_k|^{\alpha/n}\right)^q u(I_k)\right]^{1/q},$$

provided $N > (ns/q) + \alpha - n - 1$. By condition (1.2), we obtain the bound $c[\sum \lambda_k^q v(I_k)^{q/p}]^{1/q}$, which is at most $c[\sum \lambda_k^p v(I_k)]^{1/p}$ since $q/p \geqslant 1$. This gives (3.12), and so completes the proof of the sufficiency in Theorem (1.5).

If in case $\alpha = n + 2l$ we use the alternate definition of $I_\alpha f$, viz. $I_\alpha f = f * |x|^{\alpha-n} \log|x|$, then the above argument requires only minor changes. The main change occurs in the estimate for $\|b_i\|_\infty$ when $i = 0, 1$, or 2. Since $|x - y|^{\alpha-n}$ is now a polynomial, we may write, for $N \geqslant \alpha - n$,

$$|b_i(x)| = \left|\int a(y)\left\{K_i(x-y) - |x-y|^{\alpha-n} \log r\right\} dy\right|$$

$$\leqslant \int |a(y)| \left|K_i(x-y) - |x-y|^{\alpha-n} \log r\right| dy.$$

Since $b_i(x) = 0$ unless $|x| < cr$,

$$|b_i(x)| \leqslant c\int_{|x-y|<cr} |x-y|^{\alpha-n} \left|\log\left(\frac{|x-y|}{r}\right)\right| dy \leqslant cr^\alpha.$$

The argument for $i \geqslant 3$ works as before provided $N > \alpha - n - 1$ (i.e., $N \geqslant \alpha - n$ when α is an integer) since

$$D^{N+1}\left\{|x-y|^{2l}\log|x-y|\right\} = c|x-y|^{2l-N-1}$$

when $N \geqslant 2l$.

4. Necessity for Theorems (1.4) and (1.5). Let I_0 be the unit cube centered at the origin, and let a_0 be an (∞, N) atom supported in I_0 such that $I_\alpha a_0$ is not identically zero in I_0. In case $\alpha = n + 2l$, $l = 0, 1, \ldots$, we work instead with $a_0 * x_i|x|^{\alpha-n-1}$ for any fixed i. Since $I_\alpha a_0$ is continuous, there is a cube $\tilde{I}_0 \subset I_0$ and a constant $c_0 > 0$ such that $|I_\alpha a_0(x)| \geqslant c_0$ for $x \in \tilde{I}_0$. If I is any fixed cube, let L_I be the affine transformation of $\mathbf{R}^n$ such that $L_I(I_0) = I$: if I has center x_I and edgelength δ, then $L_I(x) = x_I + \delta x$. Define $a_I(x)$ by $a_I(x) = a_0(L_I^{-1}(x))$, and note that a_I is an (∞, N) atom with support in I. Moreover, if $\tilde{I}$ is the cube defined by $\tilde{I} = L_I(\tilde{I}_0)$, a change of variables easily gives $|(I_\alpha a_I)(x)| \geqslant c_0|I|^{\alpha/n}$ for $x \in \tilde{I}$. This also holds in case $\alpha = n + 2l$, $l = 0, 1, \ldots$. Thus,

$$\|I_\alpha a_I\|_{H_u^q} = \|(I_\alpha a_I)^*\|_{L_u^q} \geqslant c_0|I|^{\alpha/n}u(\tilde{I})^{1/q}.$$

Since $\|I_\alpha a_I\|_{H_u^q} \leqslant c\|a_I\|_{H_v^p}$ by assumption, we obtain

$$c_0|I|^{\alpha/n}u(\tilde{I})^{1/q} \leqslant c\|a_I\|_{H_v^p}.$$

If $0 < p \leqslant 1$ then $\|a_I\|_{H_v^p} \leqslant cv(I)^{1/p}$, and, if $p > 1$, then

$$\|a_I\|_{H_v^p} \leqslant c\|\chi_I\|_{L_v^p} = cv(I)^{1/p}$$

again. Thus, in any case, $|I|^{\alpha/n}u(\tilde{I})^{1/q} \leqslant cv(I)^{1/p}$. Finally, since $\tilde{I} \subset I$ and $|\tilde{I}|/|I| = |\tilde{I}_0|/|I_0|$ is a constant independent of I, we have $u(\tilde{I}) \approx u(I)$ by doubling, and condition (1.2) follows.

The argument is essentially the same if $I_\alpha f$ is given by the alternate definition in case $\alpha = n + 2l$. In fact, a change of variable gives

$$(I_\alpha a_I)(x) = \delta^n \int a_0(y)|x - L_I(y)|^{2l}\log|x - L_I(y)|\,dy.$$

If $x = x_I + \delta z$, $z \in \tilde{I}_0$, then

$$\log|x - L_I(y)| = \log(\delta|y - z|) = \log\delta + \log|y - z|.$$

Hence, from the above formula and the moment properties of a_0, we obtain $(I_\alpha a_I)(x) = \delta^\alpha(I_\alpha a_0)(z)$. The rest of the argument is as above.

5. Theorems (1.7) and (1.8). We begin by proving Theorem (1.7). To show (iii) note that if (1.2) holds then

$$\left(\frac{u(I)}{|I|}\right)^{1/q} \leqslant c\frac{v(I)^{1/p}|I|^{-\alpha/n}}{|I|^{1/q}} = c\left(\frac{v(I)}{|I|}\right)^{1/p}|I|^{1/p-1/q-\alpha/n}.$$

If $1/p - 1/q > \alpha/n$, the right side tends to zero a.e. as $|I| \to 0$. Thus, $u(I)/|I| \to 0$ a.e., so that $u = 0$ a.e.

It is easy to see that (1.6) implies (1.2) for $1 < p < \infty$ as follows. We have

$$\begin{aligned}|I|^{\alpha/n}u(I)^{1/q} &= |I|^{\alpha/n}u(I)^{1/p'+1/q}u(I)^{-1/p'} \\ &\leqslant c\left(\int_I u^{1/p'}v^{1/p}\,dx\right)u(I)^{-1/p'} \quad \text{by (1.1)} \\ &\leqslant c\left(\int_I u\,dx\right)^{1/p'}\left(\int_I v\,dx\right)^{1/p}u(I)^{-1/p'}\end{aligned}$$

by Hölder's inequality. Since the last expression equals $cv(I)^{1/p}$, we obtain (1.2).

Let us now show that (1.2) implies (1.6) when u and v are comparable, $1 < p < \infty$. We then have $v/u \in A_\infty(u\,dx)$, and consequently, there is a constant $c > 0$ such that for any cube I, v/u exceeds $v(I)/u(I) = \int_I (v/u)u\,dx/u(I)$ on a subset E of I with $u(E) > cu(I)$. It follows that

$$\frac{v(I)}{u(I)} \leqslant c\left(\frac{1}{u(I)}\int_I \left(\frac{v}{u}\right)^t u\,dx\right)^{1/t}$$

for any $t > 0$. Using (1.2) and then the last inequality with $t = 1/p$, we get

$$\begin{aligned}|I|^{\alpha/n}u(I)^{1/p'+1/q} &\leqslant cv(I)^{1/p}u(I)^{1/p'} = c[v(I)/u(I)]^{1/p}u(I) \\ &\leqslant c\left(\frac{1}{u(I)}\int_I v^{1/p}u^{1/p'}\,dx\right)u(I) = c\int_I v^{1/p}u^{1/p'}\,dx.\end{aligned}$$

This shows that the pair u, v satisfies (1.6), as desired.

Next, we will show that (1.2) implies (1.6) if $u = v^{q/p}$ and $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$. This is easiest to see directly, but can also be obtained by showing that u and v are then comparable. For a direct proof, note that condition (1.2) in this case is

$$\left(\frac{1}{|I|}\int_I v^{q/p}\,dx\right)^{p/q} \leqslant c\frac{1}{|I|}\int_I v\,dx,$$

i.e., $v \in \mathrm{RH}_{q/p}$, while condition (1.6) is

$$\left(\frac{1}{|I|}\int_I v^{q/p}\,dx\right)^{1/p'+1/q} \leqslant c\frac{1}{|I|}\int_I (v^{q/p})^{1/p'+1/q}\,dx. \tag{5.1}$$

To show the last inequality, note, since $v \in \mathrm{RH}_{q/p}$, that the left side is at most

$$c\left(\frac{1}{|I|}\int_I v\,dx\right)^{(q/p)(1/p'+1/q)}.$$

This in turn is less than the right side of (5.1) by Hölder's inequality since $(q/p)(1/p' + 1/q) \geqslant 1$, due to $q \geqslant p$.

To prove the same conclusion by showing that u and v are comparable if $u = v^{q/p}$, $1/q = 1/p - \alpha/n$ and (1.2) holds, note by **[11]** that $v \in \mathrm{RH}_{q/p}$ implies $v^{q/p} \in \mathrm{RH}_{1+\varepsilon}$ for some $\varepsilon > 0$. This gives

$$\left(\frac{1}{v(I)}\int_I v^{(q/p-1)r}v\,dx\right)^{1/r} \leqslant c\frac{1}{v(I)}\int_I v^{q/p}\,dx$$

for some $r > 1$, which implies $v^{q/p-1} \in A_\infty(v\,dx)$, i.e., $v/u \in A_\infty(v\,dx)$.

To complete the proof of Theorem (1.7) we must show that (1.6) implies (1.3) when $u \in D_\infty$. This will be a corollary of the following lemma, which is also useful in proving Theorem (1.8).

LEMMA (5.2). *Let u and v be measures, $a(I) \geqslant 0$ and*

$$(Mf)(x) = \sup_{I:\, x \in I} \frac{a(I)}{v(I)} \int_I |f(t)| u(t)\, dt.$$

If $v \in D_\infty$ and $1 < p \leqslant q \leqslant \infty$, then the condition $a(I)u(I)^{1/q} \leqslant cv(I)^{1/p}$ is necessary and sufficient for the weak-type estimate

$$v\{x: (Mf)(x) > \lambda\} \leqslant c\left[\|f\|_{L_u^{q'}}/\lambda\right]^{p'}, \qquad \lambda > 0,$$

where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$ and c is independent of f and λ.

PROOF. Assume that $a(I)u(I)^{1/q} \leqslant cv(I)^{1/p}$ and $v \in D_\infty$. Fix $\lambda > 0$ and let $E = \{x\colon\ (Mf)(x) > \lambda\}$. If $x \in E$ there exists I containing x such that $a(I)\int_I |f(t)|u(t)\, dt/v(I) > \lambda$. Thus,

$$\begin{aligned}
\lambda^{p'} v(I) &\leqslant \left(\frac{a(I)}{v(I)} \int_I |f(t)| u(t)\, dt\right)^{p'} v(I) \\
&\leqslant \frac{a(I)^{p'}}{v(I)^{p'-1}} \left(\int_I |f(t)|^{q'} u(t)\, dt\right)^{p'/q'} u(I)^{p'/q} \\
&= \frac{\left[a(I)u(I)^{1/q}\right]^{p'}}{v(I)^{p'-1}} \left(\int_I |f(t)|^{q'} u(t)\, dt\right)^{p'/q'} \\
&\leqslant c\frac{v(I)^{p'/p}}{v(I)^{p'-1}} \left(\int_I |f(t)|^{q'} u(t)\, dt\right)^{p'/q'} \\
&= c\left(\int_I |f(t)|^{q'} u(t)\, dt\right)^{p'/q'}
\end{aligned}$$

by using the condition and the fact that $p'/p = p' - 1$. We choose one such I for each $x \in E$ and then use the Vitali covering lemma in its simple form to pick $0 < c < \infty$ and disjoint I_k's of this type such that $v(E) \leqslant c\Sigma v(I_k)$. Here, we use the fact that $v \in D_\infty$. Thus,

$$\lambda^{p'} v(E) \leqslant c\sum \left(\int_{I_k} |f(t)|^{q'} u(t)\, dt\right)^{p'/q'} \leqslant c\left(\sum \int_{I_k} |f(t)|^{q'} u(t)\, dt\right)^{p'/q'}$$

since $p'/q' \geqslant 1$ (i.e., $p \leqslant q$). Thus, $\lambda^{p'} v(E) \leqslant c\|f\|_{L_u^{q'}}^{p'}$ as desired.

For the converse let $f = \chi_I$. If $x \in I$ we then have $(Mf)(x) \geqslant a(I)u(I)/v(I)$. Picking $\lambda = a(I)u(I)/2v(I)$, we obtain from the weak-type estimate that

$$v(I) \leqslant c\left[\frac{u(I)^{1/q'}}{a(I)u(I)/2v(I)}\right]^{p'},$$

i.e., $a(I)^{p'}u(I)^{(1-1/q')p'} \leqslant cv(I)^{p'-1}$, which is the same as $a(I)u(I)^{1/q} \leqslant cv(I)^{1/p}$. This completes the proof of the lemma.

We remark that a version of Lemma (5.2) holds for $p = 1$ if we replace the weak-type estimate by $\|Mf\|_\infty \leqslant c\|f\|_{L_u^{q'}}$. We also note that the above assumption that $v \in D_\infty$ can be dropped by using only cubes with center x when defining $(Mf)(x)$. The proof of this assertion is based on the Besicovitch covering lemma rather than the Vitali lemma. For the applications we have in mind, however, either version of the lemma ultimately leads to the same results.

In special cases the weak-type estimate given in Lemma (5.2) can be replaced by $\|Mf\|_{L_v^{p'}} \leqslant c\|f\|_{L_u^{q'}}$. We now discuss three such results and show how they lead to proofs of Theorems (1.7)(i) and (1.8).

LEMMA (5.3). *If* $1 < p \leqslant q < \infty$, $u \in D_\infty$ *and*

$$(M_1 f)(x) = \sup_{I:\, x \in I} \frac{u(I)^{1/p-1/q}}{u(I)} \int_I |f(t)|u(t)\, dt,$$

then

$$\|M_1 f\|_{L_u^{p'}} \leqslant c\|f\|_{L_u^{q'}}, \qquad 1/p + 1/p' = 1, 1/q + 1/q' = 1.$$

This result is also contained in Theorem 2 of **[13]**.

PROOF. Fix p and q with $1 < p \leqslant q < \infty$ and let $a(I) = u(I)^{1/p-1/q}$. Then $a(I)u(I)^{1/q} = u(I)^{1/p}$ and, consequently, $a(I)u(I)^{1/q\pm\varepsilon} = u(I)^{1/p\pm\varepsilon}$, $\varepsilon > 0$. Denote $1/q + \varepsilon = 1/q_1$, $1/p + \varepsilon = 1/p_1$, $1/q - \varepsilon = 1/q_2$ and $1/p - \varepsilon = 1/p_2$. Thus, if ε is small, we have $1 < p_i \leqslant q_i < \infty$ and $a(I)u(I)^{1/q_i} = u(I)^{1/p_i}$ for $i = 1, 2$ as well as $p_1 < p < p_2$ and $q_1 < q < q_2$. By Lemma (5.2), M_1 maps $L_u^{q_i'}$ into weak $L_u^{p_i'}$ for $i = 1, 2$. Hence, the desired result will follow from the Marcinkiewicz interpolation theorem if we show there exists t with $0 < t < 1$ such that $1/q' = t/q_1' + (1-t)/q_2'$ and $1/p' = t/p_1' + (1-t)/p_2'$. This amounts to requiring $1/q = t/q_1 + (1-t)/q_2$ and $1/p = t/p_1 + (1-t)/p_2$, which follow easily from the definitions of p_i and q_i if we take $t = 1/2$.

We now complete the proof of Theorem (1.7) by showing that (1.6) $\Rightarrow$ (1.3) when $u \in D_\infty$. We have for $\lambda_k > 0$,

$$\left\| \sum \lambda_k |I_k|^{\alpha/n} \chi_{I_k} \right\|_{L_u^q} = \sup \int \left(\sum \lambda_k |I_k|^{\alpha/n} \chi_{I_k} \right) gu\, dt,$$

where the sup is taken over all $g \geqslant 0$ with $\|g\|_{L_u^{q'}} = 1$. Write

$$\tag{5.4} \begin{aligned} \int \left(\sum \lambda_k |I_k|^{\alpha/n} \chi_{I_k} \right) gu\, dt &= \sum \lambda_k |I_k|^{\alpha/n} \int_{I_k} gu\, dt \\ &= \sum \lambda_k \left(\int_{I_k} v^{1/p} u^{1/p'}\, dx \right) \left(\frac{|I_k|^{\alpha/n}}{\int_{I_k} v^{1/p} u^{1/p'}\, dx} \int_{I_k} gu\, dt \right). \end{aligned}$$

By (1.6),

$$|I|^{\alpha/n} u(I)^{1/q+1/p'} \leqslant c \int_I v^{1/p} u^{1/p'}\, dx.$$

Therefore, the expression above is bounded by

$$c\sum\lambda_k\left(\int_{I_k} v^{1/p}u^{1/p'}\,dx\right)\left(\frac{u(I_k)^{1/p-1/q}}{u(I_k)}\int_{I_k} gu\,dt\right)$$
$$\leqslant c\sum\lambda_k\int_{I_k} M_1(g)v^{1/p}u^{1/p'}\,dx = c\int\left(\sum\lambda_k\chi_{I_k}\right)M_1(g)v^{1/p}u^{1/p'}\,dx.$$

Rewriting the integrand in the form $[(\Sigma\lambda_k\chi_{I_k})v^{1/p}][M_1(g)u^{1/p'}]$ and applying Hölder's inequality with exponents p and p', we get at most $c\|\Sigma\lambda_k\chi_{I_k}\|_{L^p_v}\|M_1(g)\|_{L^{p'}_u}$. Since $u \in D_\infty$, Lemma (5.3) implies $\|M_1(g)\|_{L^{p'}_u} \leqslant c\|g\|_{L^{q'}_u} = c$. Collecting estimates, we see that (u, v) satisfies (1.3), which completes the proof of Theorem (1.7).

To prove the first part of Theorem (1.8), we use the following lemma.

LEMMA (5.5). *Let* $0 \leqslant \delta < 1$ *and*

$$(M_2 f)(x) = \sup_{I:\,x\in I}\frac{v(I)^\delta}{v(I)}\int_I |f(t)|u(t)\,dt.$$

If $v \in D_\infty$, $1 < p < 1/\delta$, $q > p$ *and* $u(I) \leqslant cv(I)^{(1/p-\delta)q}$, *then* $\|M_2 f\|_{L^{p'}_v} \leqslant c\|f\|_{L^{q'}_u}$.

PROOF. Note that the condition on u and v may be written $v(I)^\delta u(I)^{1/q} \leqslant cv(I)^{1/p}$. Fix δ, p and q satisfying the hypothesis, and let $\theta = (1/p - \delta)q$. Thus, $\theta > 0$ and $u(I) \leqslant cv(I)^\theta$. Pick p_1 and p_2 close to p with $1 < p_1 < p < p_2 < 1/\delta$, and define q_1 and q_2 by $(1/p_i - \delta)q_i = \theta$, $i = 1, 2$. Note that $q_1 < q < q_2$ and $v(I)^\delta u(I)^{1/q_i} \leqslant cv(I)^{1/p_i}$ for $i = 1, 2$. We claim that $p_i \leqslant q_i$ (if the p_i are chosen close enough to p). This will use the fact that $p < q$. In fact, $p_i \leqslant q_i$ amounts to $p_i \leqslant \theta(1/p_i - \delta)^{-1}$, or to $1/p_i - \delta \leqslant \theta/p_i = (1/p - \delta)(q/p_i)$. This holds with strict inequality if $p_i = p$ since $q > p$. Thus, by continuity, it also holds if p_i is close to p. Since $v \in D_\infty$, Lemma (5.2) implies that M_2 maps $L^{q_i'}_u$ into weak $L^{p_i'}_v$ for $i = 1, 2$. By interpolation, $\|Mf\|_{L^{r'}_v} \leqslant c\|f\|_{L^{q'}_u}$ where $1/q' = t/q_1' + (1 - t)/q_2'$ and $1/r' = t/p_1' + (1 - t)/p_2'$. A simple computation shows that

$$\frac{1}{r'} = 1 - \theta\left(\frac{t}{q_1} + \frac{1-t}{q_2}\right) - \delta = 1 - \frac{\theta}{q} - \delta = 1 - \frac{1}{p} = \frac{1}{p'}.$$

Thus, $r' = p'$ and the lemma follows.

The proof of the first part of Theorem (1.8) is now similar to that of Theorem (1.7)(i). Let $1 < p < n/\alpha$, $q > p$, $v \in \mathrm{RH}_{p^*/p}$, where $1/p^* = 1/p - \alpha/n$, and (u, v) satisfy (1.2), i.e., $|I|^{\alpha/n}u(I)^{1/q} \leqslant cv(I)^{1/p}$. We must show that (u, v) satisfies (1.3). As before (see (5.4)), it is enough to show that

$$\sum\lambda_k|I_k|^{\alpha/n}\int_{I_k} gu\,dt \leqslant c\left\|\sum\lambda_k\chi_{I_k}\right\|_{L^p_v} \tag{5.6}$$

for $\lambda_k \geqslant 0$, $g \geqslant 0$ and $\|g\|_{L^{q'}_u} = 1$. Note by (1.2) and Hölder's inequality that

$$u(I)^{1/q} \leqslant \frac{cv(I)^{1/p}}{|I|^{\alpha/n}} \leqslant c\left(\int_I v^{p^*/p}\,dx\right)^{1/p^*}.$$

Thus, letting $w = v^{p^*/p}$, we have

$$u(I) \leqslant cw(I)^{q/p^*} = cw(I)^{(1/p-\alpha/n)q}. \tag{5.7}$$

The left side of (5.6) equals

$$\sum \lambda_k \left(\int_{I_k} v^{1/p} w^{1/p'}\, dx \right) \left(\frac{|I_k|^{\alpha/n}}{\int_{I_k} v^{1/p} w^{1/p'}\, dx} \int_{I_k} gu\, dt \right).$$

Since $v^{1/p} w^{1/p'} = w^{1/p^* + 1/p'} = w^{1-\alpha/n}$,

$$\frac{|I|^{\alpha/n}}{\int_I v^{1/p} w^{1/p'}\, dx} = \frac{|I|^{\alpha/n}}{\int_I w^{1-\alpha/n}\, dx}. \tag{5.8}$$

However, the fact that $v \in \mathrm{RH}_{p^*/p}$ is equivalent to $w \in A_\infty$ (see, e.g., Corollary (6.2)), which implies

$$\left(w(I)/|I| \right)^{1-\alpha/n} \leqslant c \int_I w^{1-\alpha/n}\, dx / |I|.$$

Hence, (5.8) is bounded by $cw(I)^{\alpha/n-1}$ and, consequently, the left side of (5.6) is at most

$$c \sum \lambda_k \left(\int_{I_k} v^{1/p} w^{1/p'}\, dx \right) \left(\frac{w(I)^{\alpha/n}}{w(I)} \int_I gu\, dt \right).$$

If we define

$$(\tilde{M}_2 g)(x) = \sup_{I:\, x \in I} \frac{w(I)^{\alpha/n}}{w(I)} \int_I gu\, dt,$$

we get at most

$$\begin{aligned} c \sum \lambda_k \int_{I_k} (\tilde{M}_2 g) v^{1/p} w^{1/p'}\, dx &= c \int \left(\sum \lambda_k \chi_{I_k} \right) (\tilde{M}_2 g) v^{1/p} w^{1/p'}\, dx \\ &\leqslant c \left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_v} \|\tilde{M}_2 g\|_{L^{p'}_w}. \end{aligned}$$

Now apply Lemma (5.5) with δ, u and v there taken to be α/n, u and w, respectively. Note that $w \in D_\infty$ since $w \in A_\infty$, and that (5.7) is the condition required of u and v in Lemma (5.5). Thus, $\|\tilde{M}_2 g\|_{L^{p'}_w} \leqslant c\|g\|_{L^{q'}_u} = c$, and (5.6) follows. This completes the proof of the first part of Theorem (1.8).

Finally, to prove the second part we use the next lemma.

LEMMA (5.9). *Let*

$$(M_3 f)(x) = \sup_{I:\, x \in I} \frac{u(I)^\delta}{v(I)} \int_I |f(t)| u(t)\, dt.$$

If $v \in D_\infty$, $\delta > -1/q$, $u(I)^{p(\delta+1/q)} \leqslant cv(I)$ and $1 < p < q$, then

$$\|M_3 f\|_{L^{p'}_v} \leqslant c\|f\|_{L^{q'}_u}.$$

PROOF. The proof is nearly the same as that of Lemma (5.5). Note that the condition on u and v may be written $u(I)^\delta u(I)^{1/q} \leqslant cv(I)^{1/p}$. Fix δ, p and q, and let $\theta = (1/p)(\delta + 1/q)^{-1}$, so that $\theta > 0$ and $u(I) \leqslant cv(I)^\theta$. Pick p_1 and p_2 with $1 < p_1 < p < p_2$, and define q_1 and q_2 by $1/p_i = \theta(\delta + 1/q_i)$. If p_1 and p_2 are close to p, we have $q_1 < q < q_2$, $u(I)^{\delta}u(I)^{1/q_i} \leqslant cv(I)^{1/p_i}$ and $p_i \leqslant q_i$: the fact that $p_i \leqslant q_i$ amounts to $1/q_i \leqslant 1/p_i$, or to $(p/p_i)(\delta + 1/q) - \delta \leqslant 1/p_i$, which is true for p_i near p since $q > p$. Thus, by Lemma (5.2), M_3 maps $L_u^{q_i'}$ into weak $L_v^{p_i'}$, $i = 1, 2$. The conclusion of the lemma now follows as usual by interpolation.

To show that (1.2) implies (1.3) if $1 < p < q < \infty$, $v \in D_\infty$ and $u \in A_\infty$, note by Hölder's inequality and (1.2) that

$$\left(\int_I u^{1/(\alpha q/n+1)}\,dx\right)^{\alpha/n+1/q} \leqslant |I|^{\alpha/n}u(I)^{1/q} \leqslant cv(I)^{1/p}.$$

Hence, letting $w = u^{1/(\alpha q/n+1)}$, we have $w(I)^{\alpha/n+1/q} \leqslant cv(I)^{1/p}$. The idea of the proof is to take advantage of this fact. It is enough to show (5.6) for g with $\|g\|_{L_u^{q'}} = 1$. If $g_1 = gw^{\alpha q/nq'}$, the left side of (5.6) equals

$$\begin{aligned}&\sum\lambda_k|I_k|^{\alpha/n}\int_{I_k} g_1 w^{1+\alpha/n}\,dt\\&\qquad\leqslant \sum\lambda_k v(I_k)\frac{|I_k|^{\alpha/n}}{v(I_k)}\left(\int_{I_k} w^{1+\alpha r'/n}\,dt\right)^{1/r'}\left(\int_{I_k} g_1^r w\,dt\right)^{1/r}\end{aligned}$$

by Hölder's inequality, for r to be chosen close to q' with $1 < r < q'$. Assuming for the moment that

$$\frac{|I|^{\alpha/n}}{v(I)}\left(\int_I w^{1+\alpha r'/n}\,dt\right)^{1/r'} \leqslant c\frac{w(I)^{\delta/r}}{v(I)^{1/r}}, \qquad \delta \text{ to be chosen,} \tag{5.10}$$

we get the bound

$$c\sum\lambda_k v(I_k)\left(\frac{w(I_k)^{\delta}}{v(I_k)}\int_{I_k} g_1^r w\,dt\right)^{1/r} \leqslant c\int\left(\sum\lambda_k\chi_{I_k}\right)(\tilde{M}_3(g_1^r))^{1/r}v(x)\,dx,$$

where

$$\tilde{M}_3(f)(x) = \sup_{I:\,x\in I}\frac{w(I)^{\delta}}{v(I)}\int_I |f(t)|w(t)\,dt.$$

By Hölder's inequality we obtain at most

$$c\left\|\sum\lambda_k\chi_{I_k}\right\|_{L_v^p}\|\tilde{M}_3(g_1^r)\|_{L_v^{p'/r}}^{1/r}.$$

From Lemma (5.9) we have

$$\|\tilde{M}_3(g_1^r)\|_{L_v^{p'/r}} \leqslant c\|g_1^r\|_{L_w^{q'/r}} = c\|g\|_{L_u^{q'}}^r = c,$$

and so we will be done, provided that $\delta > -1/(q'/r)'$, $1 < (p'/r)' < (q'/r)'$ and

$$w(I)^{\delta+1/(q'/r)'} \leqslant cv(I)^{1/(p'/r)'}. \tag{5.11}$$

If we choose δ so that $\delta + 1/(q'/r)' = p(\alpha/n + 1/q)/(p'/r)'$, (5.11) follows from our earlier observation that $w(I)^{\alpha/n+1/q} \leqslant cv(I)^{1/p}$. This choice of δ also ensures that $\delta > -1/(q'/r)'$. The inequalities $1 < (p'/r)' < (q'/r)'$ follow from $r < q'$ and

$p < q$. Thus, it remains only to verify (5.10). If we rewrite (5.10) in terms of u, it amounts to

$$|I|^{\alpha/n}\left(\int_I u^{(\alpha r'/n+1)/(\alpha q/n+1)}\,dx\right)^{1/r'} \leqslant c\left(\int_I u^{1/(\alpha q/n+1)}\,dx\right)^{\delta/r} v(I)^{1/r'}.$$

Note that $(\alpha r'/n + 1)/(\alpha q/n + 1) = 1 + \varepsilon$ for $\varepsilon > 0$ and arbitrarily small if r is close to q', $r < q'$. Since $u \in A_\infty$,

$$\int_I u^{1+\varepsilon}\,dx \leqslant c|I|^{-\varepsilon}\left(\int_I u\,dx\right)^{1+\varepsilon}$$

and

$$\int_I u^{1/(\alpha q/n+1)}\,dx \geqslant c|I|^{(\alpha q/n)/(\alpha q/n+1)}\left(\int_I u\,dx\right)^{1/(\alpha q/n+1)}.$$

Thus, (5.10) will follow if

$$|I|^{\alpha/n-\varepsilon/r'}u(I)^{(1+\varepsilon)/r'} \leqslant c\left[|I|^{(\alpha q/n)/(\alpha q/n+1)}u(I)^{1/(\alpha q/n+1)}\right]^{\delta/r} v(I)^{1/r'}.$$

This is just condition (1.2), as can be seen by computing exponents. This completes the proof of Theorem (1.8).

6. An example. Theorems (1.9) and (1.10). In this section we prove Theorem (1.9). Also, as indicated in the Introduction, we obtain Theorem (1.10) as a corollary of the proof.

We need several lemmas about weight functions.

LEMMA (6.1). *If $u \in A_\infty$ then $u^s \in \mathrm{RH}_{1/s}$ for $0 < s < 1$. Conversely, if $u^s \in \mathrm{RS}_{1/s}$ for some s with $0 < s < 1$, then $u \in A_\infty$.*

PROOF. If $u \in A_\infty$, then by [3], $u(x) \geqslant c|I|^{-1}\int_I u\,dy$ for x in a subset of I of measure proportional to $|I|$. Thus, by raising both sides to the power s and integrating, we get

$$\left(\frac{1}{|I|}\int_I u\,dy\right)^s \leqslant c\frac{1}{|I|}\int_I u^s\,dx.$$

This amounts to $u^s \in \mathrm{RH}_{1/s}$, $0 < s < 1$.

Conversely, if $u^s \in \mathrm{RH}_{1/s}$ for some s, $0 < s < 1$, then by [3], $u^s \in \mathrm{RH}_{(1+\varepsilon)/s}$ for some $\varepsilon > 0$. Thus,

$$\left(\frac{1}{|I|}\int_I u^{1+\varepsilon}\,dy\right)^{s/(1+\varepsilon)} \leqslant c\frac{1}{|I|}\int_I u^s\,dy.$$

By Hölder's inequality the right side is less than $c(|I|^{-1}\int_I u\,dy)^s$. Therefore, $u \in \mathrm{RH}_{1+\varepsilon}$, so $u \in A_\infty$.

COROLLARY (6.2). *Let $r > 1$. Then $w^r \in A_\infty$ if and only if $w \in \mathrm{RH}_r$.*

To see this, just apply the lemma with $u = w^r$ and $s = 1/r$. In particular, note that if $q > p$ then the conditions $w^{q/p} \in A_\infty$ and $w \in \mathrm{RH}_{q/p}$ are the same.

LEMMA (6.3). *If $w \in D_\infty$ there exists $\varepsilon = \varepsilon(w) > 0$ such that*

$$w(x)|x|^\gamma,\ w(x)\big(|x|/(1+|x|)\big)^\gamma \in D_\infty \quad \textit{if } \gamma > -\varepsilon.$$

PROOF. Let $d = \operatorname{dist}(I, 0)$. If the edgelength h of I satisfies $h \leqslant \frac{1}{2}d$, then $|x| \approx d$ on $2I$. Thus,

$$\int_{2I} w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx \approx \left(\frac{d}{1+d}\right)^\gamma \int_{2I} w\,dx \approx \left(\frac{d}{1+d}\right)^\gamma \int_I w\,dx$$

$$\approx \int_I w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx$$

for any γ. The proof in this case for $w(x)|x|^\gamma$ is similar.

If $h > \frac{1}{2}d$, then by subdivision of I, select a cube $\tilde{I} \subset I$ so that the edgelength $\tilde{h}$ of $\tilde{I}$ satisfies $\tilde{h} = h/8$ and $\tilde{h} < \frac{1}{2}\operatorname{dist}(\tilde{I}, 0)$. Let I^* be a cube with center 0 such that $2I \subset I^*$ and the edgelength h^* of I^* satisfies $h^* \leqslant 8h$. Then

$$\int_{2I} w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx \leqslant \int_{I^*} w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx$$

$$\leqslant c\sum\left(\int_{I^*:\,|x|\approx 2^{-k}h^*} w(x)\,dx\right)\left(\frac{2^{-k}h^*}{1+2^{-k}h^*}\right)^\gamma$$

$$\leqslant c\sum 2^{-k\varepsilon}\left(\frac{2^{-k}h^*}{1+2^{-k}h^*}\right)^\gamma \int_{I^*} w(x)\,dx,$$

since $w \in \mathrm{RD}_{\varepsilon/n}$ for some $\varepsilon > 0$ (see §2). Now, since $w \in D_\infty$ and $|x| \approx \tilde{h} \approx h$ on $\tilde{I}$,

$$\int_{I^*} w\,dx \leqslant c\int_{\tilde{I}} w\,dx \leqslant c\left(\frac{h}{1+h}\right)^{-\gamma} \int_{\tilde{I}} w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx$$

$$\leqslant c\left(\frac{h}{1+h}\right)^{-\gamma} \int_I w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx.$$

Substituting this estimate in the sum above and recalling that $h^* \approx h$, we will obtain $w(x)(|x|/(1+|x|))^\gamma \in D_\infty$ by showing

$$\sum 2^{-k\varepsilon}\left(\frac{2^{-k}h}{1+2^{-k}h}\right)^\gamma \leqslant c\left(\frac{h}{1+h}\right)^\gamma. \tag{6.4}$$

If $h \leqslant 1$ this reduces to $\sum 2^{-k\varepsilon}(2^{-k}h)^\gamma \leqslant ch^\gamma$, which is valid if $\gamma > -\varepsilon$. If $h > 1$ the left side of (6.4) is at most

$$\sum_{2^{-k}h<1} 2^{-k\varepsilon}(2^{-k}h)^\gamma + \sum_{2^{-k}h\geqslant 1} 2^{-k\varepsilon} \leqslant ch^\gamma h^{-\varepsilon-\gamma} + c \leqslant c,$$

since $h > 1$ and $\varepsilon > 0$. In this case the right side of (6.4) exceeds a fixed positive constant, and (6.4) follows. This completes the proof for $w(x)(|x|/(1+|x|))^\gamma$. For $w(x)|x|^\gamma$, the argument in case $h > \frac{1}{2}d$ is similar but simpler. The analogue of (6.4) in this case is $\sum 2^{-k\varepsilon}(2^{-k}h)^\gamma \leqslant ch^\gamma$, which is valid if $\gamma > -\varepsilon$.

LEMMA (6.5). *If $w \in A_\infty$ there exists $\varepsilon = \varepsilon(w) > 0$ such that*

$$w(x)|x|^\gamma,\ w(x)\big(|x|/(1+|x|)\big)^\gamma \in A_\infty \quad \textit{if } \gamma > -\varepsilon.$$

PROOF. Given a cube I, we chose a subcube $\tilde{I}$ of I such that $|\tilde{I}| \geqslant c|I|$ for some $c > 0$, $0 \notin \tilde{I}$, and the edgelength of $\tilde{I}$ is less than $\frac{1}{2}\operatorname{dist}(\tilde{I}, 0)$. Letting $\delta = \operatorname{dist}(\tilde{I}, 0)$, we see that $|x| \approx \delta$ on $\tilde{I}$. By the previous lemma there exists $\varepsilon > 0$ such that $w(x)(|x|/(1 + |x|))^\gamma \in D_\infty$ for $\gamma > -\varepsilon$. Thus, for such γ,

$$\frac{1}{|I|}\int_I w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx \leqslant c\frac{1}{|\tilde{I}|}\int_{\tilde{I}} w(x)\left(\frac{|x|}{1+|x|}\right)^\gamma dx$$

$$\leqslant c\left(\frac{\delta}{1+\delta}\right)^\gamma \frac{1}{|\tilde{I}|}\int_{\tilde{I}} w(x)\,dx.$$

Since $w \in A_\infty$ there is a subset E of $\tilde{I}$ such that $|E| > c|\tilde{I}|$ and $|\tilde{I}|^{-1}\int_{\tilde{I}} w(x)\,dx \leqslant cw(z)$ for $z \in E$. Thus, $|E| > c|I|$, and since $|z| \approx \delta$ on E (in fact, on $\tilde{I}$),

$$\left(\frac{\delta}{1+\delta}\right)^\gamma \frac{1}{|\tilde{I}|}\int_{\tilde{I}} w(x)\,dx \leqslant c\left(\frac{|z|}{1+|z|}\right)^\gamma w(z), \qquad z \in E.$$

This proves that $w(x)(|x|/(1 + |x|))^\gamma \in A_\infty$ for $\gamma > -\varepsilon$. The argument for $w(x)|x|^\gamma$ is similar.

LEMMA (6.6). *Let $w \in D_\infty$ and $\{a_k\}_{k=1}^m$ be a sequence of distinct points in $\mathbf{R}^n$. If $0 \leqslant \beta_k \leqslant \beta$ for $k = 1, \ldots, m$, there exists $c > 0$ independent of I such that*

$$\int_I w(x)(1+|x|)^\beta \prod_1^m \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\beta_k} dx \geqslant c|I|^{\beta/n}\int_I w(x)\,dx. \tag{6.7}$$

In particular, taking just one term, we see that

$$\int_I w(x)|x-a|^\beta\,dx \geqslant c|I|^{\beta/n}\int_I w(x)\,dx, \qquad \beta \geqslant 0.$$

PROOF. We consider the case when there is more than one a_k; the case when there is only one a_k is similar but simpler. Let $\rho = \min_{j \neq k}|a_j - a_k|$. Then $\rho > 0$ and there exists $c_0 > 0$ depending on ρ and the number of a_k's such that any cube I contains a subcube $\tilde{I}$ with $|\tilde{I}| > c_0|I|$ and $a_k \notin \tilde{I}$ for all k. Since the integral on the left of (6.7) decreases when I is replaced by $\tilde{I}$, and since $w \in D_\infty$, it is enough to prove the lemma for $\tilde{I}$. Thus, we may assume that I contains none of the a_k's. Furthermore, if $d = \min_k \operatorname{dist}(I, a_k)$ and h is the diameter of I, we may assume $h < \frac{1}{2}d$ by subdividing I into $(6n)^n$ equal subcubes and considering one which contains the center of I. Let a_{k_0} be an a_k closest to I, so that $d = \operatorname{dist}(I, a_{k_0})$ and $d \leqslant |x - a_{k_0}| \leqslant 3d/2$ for $x \in I$. We claim that there exists $c_0 > 0$ depending only on ρ such that $|x - a_k| \geqslant c_0$ for $x \in I$ and $k \neq k_0$. To see this, consider the cases $d \geqslant \rho/3$ and $d < \rho/3$. If $d \geqslant \rho/3$ the claim is true for all k (including $k = k_0$) with $c_0 = \rho/3$ since $|x - a_k| \geqslant |x - a_{k_0}| \geqslant d$ if $x \in I$. If $d < \rho/3$, $x \in I$ and $k \neq k_0$, then

$$|x-a_k| \geqslant |a_k - a_{k_0}| - |a_{k_0} - x| \geqslant \rho - 3d/2 > \rho - \rho/2 = \rho/2,$$

and the claim follows. Thus, for such I, the left side of (6.7) equals

$$\int_I w(x)(1+|x|)^\beta \left(\frac{|x-a_{k_0}|}{1+|x-a_{k_0}|}\right)^{\beta_{k_0}} \prod_{k\neq k_0}\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\beta_k} dx$$

$$\geqslant \prod_{k\neq k_0}\left(\frac{c_0}{1+c_0}\right)^{\beta_k}\int_I w(x)(1+|x|)^\beta\left(\frac{|x-a_{k_0}|}{1+|x-a_{k_0}|}\right)^{\beta_{k_0}} dx$$

since $\beta_k \geqslant 0$. Since $\beta_{k_0} \leqslant \beta$, this exceeds a positive constant times

$$\int_I w(x)(1+|x|)^\beta\left(1+|x-a_{k_0}|\right)^{-\beta}|x-a_{k_0}|^\beta\, dx \geqslant c\int_I w(x)|x-a_{k_0}|^\beta\, dx$$

$$\geqslant c|I|^{\beta/n}\int_I w(x)\, dx,$$

$c > 0$, since $|x-a_{k_0}| \geqslant d > 2h$ for $x \in I$. The proof of the lemma is now complete.

PROOFS OF THEOREMS (1.9) AND (1.10). We first prove Theorem (1.9). Thus, let $1 < p < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$, $\beta/n = \alpha/n - (1/p - 1/q)$,

$$\Pi(x) = (1+|x|)^M \prod_{k=1}^m \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\mu_k}, \qquad \mu_k > 0,\, M \geqslant \beta,$$

and

$$\Pi_\beta(x) = (1+|x|)^{M-\beta}\prod_{k=1}^m\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\gamma_k}, \qquad \gamma_k = \max\{\mu_k - \beta, 0\},$$

where $\{a_k\}_{k=1}^m$ are distinct points of $\mathbf{R}^n$. We will show that the pair $(u, v) = (\Pi_\beta^q w^{q/p}, \Pi^p w)$ satisfies condition (1.6) and $u, v \in D_\infty$ provided that $w \in D_\infty$ when $q = p$ and $w \in \mathrm{RH}_{q/p}$ when $q > p$.

Since μ_k, γ_k, M, $M - \beta \geqslant 0$ and $(1+|x|)^M \approx 1 + |x|^M$, $(1+|x|)^{M-\beta} \approx 1 + |x|^{M-\beta}$, it follows by repeated application with translation of Lemma (6.3) that u, $v \in D_\infty$. Furthermore, if $p \neq q$, then $w^{q/p} \in A_\infty$ by Corollary (6.2), and $u, v \in A_\infty$ by Lemma (6.5). To check (1.6) we must show that

$$|I|^{\alpha/n}\left(\int_I u\, dx\right)^{1/p'+1/q} \leqslant c\int_I u^{1/p'}v^{1/p}\, dx. \tag{6.8}$$

Let

$$\Pi^*(x) = \frac{\Pi(x)}{\Pi_\beta(x)} = (1+|x|)^\beta \prod_{k=1}^m\left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\mu_k-\gamma_k},$$

and note that $\mu_k - \gamma_k = \min(\mu_k, \beta)$. Thus, $0 \leqslant \mu_k - \gamma_k \leqslant \beta$. Write the integral on the right side of (6.8) as

$$\int_I u^{1/p'+1/q}u^{-1/q}v^{1/p}\, dx = \int_I u^{1/p'+1/q}\Pi^*\, dx \tag{6.9}$$

since $u^{-1/q}v^{1/p} = \Pi^*$. Note that $u^{1/p'+1/q} \in D_\infty$: if $q = p$, we know $u \in D_\infty$, and if $q \neq p$, then $1/p' + 1/q < 1$, and the fact that $u \in A_\infty$, together with Lemma (6.1), clearly implies $u^{1/p'+1/q} \in D_\infty$ (even A_∞). Hence, by Lemma (6.6),

$$\int_I u^{1/p'+1/q}\Pi^* \, dx \geqslant c|I|^{\beta/n} \int_I u^{1/p'+1/q} \, dx.$$

Since

$$\frac{\alpha}{n} - \frac{\beta}{n} = \frac{1}{p} - \frac{1}{q} = 1 - \left(\frac{1}{p'} + \frac{1}{q}\right),$$

condition (6.8) will then follow from

$$\left(\frac{1}{|I|}\int_I u \, dx\right)^{1/p'+1/q} \leqslant c\frac{1}{|I|}\int_I u^{1/p'+1/q} \, dx.$$

This is obvious when $q = p$, and when $q > p$ amounts to the statement that $u^s \in \mathrm{RH}_{1/s}$ with $s = 1/p' + 1/q$, which follows by Lemma (6.1). This completes the proof of Theorem (1.9).

To prove Theorem (1.10) let $0 < p < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha/n$, $\beta/n = \alpha/n - (1/p - 1/q)$ and

$$\Pi^*(x) = (1 + |x|)^\beta \prod_{k=1}^m \left(\frac{|x - a_k|}{1 + |x - a_k|}\right)^{\nu_k}, \qquad 0 \leqslant \nu_k \leqslant \beta,$$

where $\{a_k\}_{k=1}^m$ are distinct points of $\mathbf{R}^n$. We will show that the pair $(u, v) = ([v^{1/p}/\Pi^*]^q, v)$ satisfies $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{H_v^p}$ if u, $v \in D_\infty$ and, in addition, when $p \neq q$, $u \in A_\infty$. In case $1 < p < \infty$, we follow the above proof beginning with (6.9) to see that (u, v) satisfies (1.6). If $0 < p \leqslant 1$, the condition to be checked instead is (1.2), i.e., $|I|^{\alpha/n}u(I)^{1/q} \leqslant cv(I)^{1/p}$, which becomes

$$|I|^{\alpha/n}\left(\int_I u \, dx\right)^{1/q} \leqslant c\left(\int_I u^{p/q}\Pi^{*p} \, dx\right)^{1/p}.$$

The right side exceeds $c|I|^{\beta/n}(\int_I u^{p/q} \, dx)^{1/p}$ by Lemma (6.6) if $u^{p/q} \in D_\infty$, which is true by hypothesis if $q = p$, and for $q > p$ it follows from the fact that $u \in A_\infty$ by Lemma (6.1). The desired estimate is now immediate if $q = p$, and if $q > p$ it follows by using $u^{p/q} \in \mathrm{RH}_{q/p}$, i.e, $u \in A_\infty$.

7. Proof of Theorem (1.11). To show that condition (1.3) is necessary for the inequality $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ with $I_\alpha f = f * |x|^{\alpha - n}$, $0 < \alpha < n$, let $f = \sum \lambda_k \chi_{I_k}$, $\lambda_k > 0$. Then

$$\begin{aligned}(I_\alpha f)(x) &= \sum \lambda_k \int_{I_k} |x - y|^{\alpha - n} \, dy \geqslant \sum \lambda_k \left(\int_{I_k} |x - y|^{\alpha - n} \, dy\right)\chi_{I_k}(x) \\ &\geqslant \sum \lambda_k \left(c|I_k|^{\alpha/n}\right)\chi_{I_k}(x),\end{aligned}$$

since if $x \in I_k$ then $|x - y| \approx \operatorname{diam} I_k$ for y in a subset of I_k with measure proportional to $|I_k|$. Condition (1.3) now follows immediately from the norm inequality.

The above argument does not require $\alpha < n$, but we note that condition (1.3), or even (1.2), is not compatible with $v \in A_p$ if $\alpha \geqslant n$. In fact, for large I centered at 0 and $\alpha \geqslant n$, (1.2) gives

$$|I| \leqslant |I|^{\alpha/n} \leqslant c|I|^{\alpha/n} u(I)^{1/q} \leqslant cv(I)^{1/p},$$

i.e., $v(I) \geqslant c|I|^p$. However, any $v \in A_p$, $p > 1$, satisfies $v \in D_{p-\varepsilon}$ for some $\varepsilon > 0$, which contradicts the previous statement.

The proof of the sufficiency part of Theorem (1.11) is based on some slight modifications in the proof given in §3 and on an approximation argument. We first show that (1.3), together with the fact that $v \in D_\infty$, implies

$$\text{(7.1)} \qquad \int_{\mathbf{R}^n} \frac{u(x)}{(1+|x|)^L} dx < \infty \quad \text{for some } L.$$

In fact, (7.1) holds if we just assume (1.2) and $v \in D_\infty$. To see this, let J_l be the cube with center 0 and edgelength 2^l, $l = 0, 1, \ldots$. By doubling we have $v(J_l) \leqslant c2^{lnt}v(J_0) = c2^{lnt}$ for some $t \geqslant 1$. Thus, by (1.2), $u(J_l) \leqslant c2^{l((nt/p)-\alpha)q}$, and it follows easily that, for $\varepsilon > 0$,

$$\int_{\mathbf{R}^n} \frac{u(x)}{(1+|x|)^{((nt/p)-\alpha+\varepsilon)q}} dx < \infty,$$

as desired.

We next claim that (1.3), together with $v \in A_p$, implies

$$\text{(7.2)} \qquad \int_{|x|>1} \frac{v(x)^{-1/(p-1)}}{|x|^{(n-\alpha)p'}} dx < \infty.$$

To see this, let J_l be as above and split $J_l - J_{l-1}$ for $l \geqslant 1$ into $c(n)$ nonoverlapping cubes with edgelength 2^{l-2}. Thus, we decompose $\mathbf{R}^n \setminus J_0$ into nonoverlapping cubes $\{I_k\}_{k=1}^\infty$ such that $\operatorname{dist}(I_k, 0) \approx |I_k|^{1/n}$ and $\bar{I}_k = 10I_k$ contains J_0. Using (1.3) and Lemma (2.3) we then get, for $\lambda_k > 0$,

$$\left(\sum_1^M \lambda_k |\bar{I}_k|^{\alpha/n}\right) u(J_0)^{1/q} \leqslant c \left\| \sum_1^M \lambda_k |\bar{I}_k|^{\alpha/n} \chi_{\bar{I}_k} \right\|_{L_u^q} \leqslant c \left\| \sum_1^M \lambda_k \chi_{\bar{I}_k} \right\|_{L_v^p}$$
$$\leqslant c \left\| \sum_1^M \lambda_k \chi_{I_k} \right\|_{L_v^p} = c \left(\sum_1^M \lambda_k^p v(I_k) \right)^{1/p}.$$

Now pick λ_k so that $\lambda_k^p v(I_k) = \lambda_k |\bar{I}_k|^{\alpha/n}$, i.e.,

$$\lambda_k = \left(|\bar{I}_k|^{\alpha/n} / v(I_k)\right)^{1/(p-1)} = c\left(|I_k|^{\alpha/n-1}|I_k|/v(I_k)\right)^{1/(p-1)}.$$

Since $v \in A_p$,

$$\lambda_k \approx |I_k|^{(\alpha/n-1)/(p-1)} \int_{I_k} v^{-1/(p-1)}\, dx / |I_k|.$$

We get

$$\left(\sum_1^M \lambda_k |\bar{I}_k|^{\alpha/n}\right) u(J_0)^{1/q} \leqslant c \left(\sum_1^M \lambda_k |\bar{I}_k|^{\alpha/n}\right)^{1/p},$$

and therefore $\sum_1^M \lambda_k |\bar{I}_k|^{\alpha/n} \leqslant cu(J_0)^{-p'/q}$. But also

$$\sum_1^M \lambda_k \left|\bar{I}_k\right|^{\alpha/n} \approx \sum_1^M \left|I_k\right|^{p'(\alpha/n-1)} \int_{I_k} v^{-1/(p-1)}\,dx$$

$$\approx \sum_1^M \int_{I_k} \frac{v^{-1/(p-1)}}{|x|^{p'(n-\alpha)}}\,dx$$

since $|I_k| \approx |x|^n$ for $x \in I_k$. Hence,

$$\int_{\bigcup_1^M I_k} \frac{v^{-1/(p-1)}}{|x|^{p'(n-\alpha)}}\,dx \leqslant cu(J_0)^{p'/q}.$$

If we let $M \to \infty$ we obtain (7.2).

The next step is to show that if $v \in A_p$ and (1.3) holds then $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ if f is (a multiple of) an (∞, N) atom. Since $v \in A_p$, L_v^p and H_v^p coincide, and it is enough to show that $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{H_v^p}$ for such f. This can be proved, without assuming that $u \in D_\infty$, by making a small change in the argument in §3 for $p > 1$. Follow the proof there leading to (3.2), and note that

$$|I_\alpha f_M| \leqslant \sum_{i,k} \lambda_k \mu_{k,i} \left|\frac{b_{k,i}}{\mu_{k,i}}\right| \leqslant \sum_{i,k} \lambda_k \mu_{k,i} \chi_{c2^i I_k}.$$

Then, instead of applying Lemma (2.1), simply use Minkowski's inequality to obtain

$$\|I_\alpha f_M\|_{L_u^q} \leqslant \sum_i \left\| \sum_k \lambda_k \mu_{k,i} \chi_{c2^i I_k} \right\|_{L_u^q}.$$

This leads as before to an analogue of (3.2) with $\|\cdot\|_{L_u^q}$ on the left rather than $\|\cdot\|_{H_u^q}$. The remainder of the argument in §3 then gives the desired result; the fact that $I_\alpha f_M \to I_\alpha f$ in L_u^q was shown in the course of the argument. Note that (3.11) now holds by (7.1).

It remains only to show that $\|I_\alpha f\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$ holds for general f, with $I_\alpha f$ defined as usual, knowing that it holds when f is any multiple of an (∞, N) atom. For this we will give a simple approximation argument based on (7.2) and the fact that $v^{-1/(p-1)} \in L_{\text{loc}}^1$. Since multiples of atoms are dense in H_v^p, they are also dense in L_v^p. If $\bar{I}_\alpha$ denotes the extension by continuity of I_α to all of L_v^p, it is enough to show that $I_\alpha f$, defined as usual, exists and equals $\bar{I}_\alpha f$ for all $f \in L_v^p$. For this it suffices to show that $I_\alpha f$ exists, and that if $f_j \to f$ in L_v^p then, for each $R > 0$, there is a subsequence j_k such that $I_\alpha f_{j_k} \to I_\alpha f$ a.e. in $|x| < R$.

If $|x| < R$,

$$(7.3)\qquad |I_\alpha f(x)| \leqslant \int_{|y|<2R} |f(y)|\,|x-y|^{\alpha-n}\,dy + c\int_{|y|>2R} |f(y)|\,|y|^{\alpha-n}\,dy.$$

For the first term on the right, Fubini's theorem gives

$$\left\| \int_{|y|<2R} |f(y)|\,|x-y|^{\alpha-n}\,dy \right\|_{L^1(|x|<R)} \leqslant \int_{|y|<2R} |f(y)|\,dy \cdot \int_{|z|<3R} |z|^{\alpha-n}\,dz$$

$$= cR^\alpha \|f\|_{L^1(|x|<2R)}.$$

Note that $f \in L^1_{\mathrm{loc}}$: in fact, by Hölder's inequality,

$$\|f\|_{L^1(|x|<2R)} \leqslant \|f\|_{L^p_v} \left(\int_{|x|<2R} v(x)^{-1/(p-1)}\, dx \right)^{1/p'} = c_{v,R} \|f\|_{L^p_v}.$$

Thus, the $L^1(|x| < R)$ norm of the first term on the right of (7.3) is less than $c_{v,R}\|f\|_{L^p_v}$, and so this term is finite a.e. in $|x| < R$. By Hölder's inequality, the second term on the right of (7.3) is at most

$$c\|f\|_{L^p_v} \left(\int_{|y|>2R} v(y)^{-1/(p-1)} |y|^{(\alpha-n)p'}\, dy \right)^{1/p'},$$

which equals $c_{v,R}\|f\|_{L^p_v}$ by (7.2). Combining facts we see that $I_\alpha f$ converges absolutely a.e. in $|x| < R$ and so a.e. in $\mathbf{R}^n$. Moreover, applying the same argument with f replaced by $f - f_j$, where $\|f - f_j\|_{L^p_v} \to 0$, we see that $I_\alpha f - I_\alpha f_j$ $(= I_\alpha(f - f_j))$ $\to 0$ in the $L^1(|x| < R)$ norm. Hence, there is a subsequence with the desired property. This completes the proof.

8. Remark. We would like to show how the results of [**10** and **12**] are related to our theorems. The necessity statement in the following theorem is the main result in [**10**].

THEOREM (8.1). *Let $0 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then $v \in \mathrm{RH}_{q/p}$ if and only if*

$$\|I_\alpha f\|_{H^q_{v^{q/p}}} \leqslant c\|f\|_{H^p_v}$$

when f is any (∞, N) atom.

In case $\alpha = n + 2l$, $l = 0, 1, \ldots$, $I_\alpha f$ is interpreted as usual as $f * x_i|x|^{\alpha-n-1}$ for any i or, alternately, as $f * |x|^{\alpha-n}\log|x|$. It is easy to obtain the theorem from our results. In fact, the condition $v \in \mathrm{RH}_{q/p}$ is equivalent to condition (1.2) for the pair $(v^{q/p}, v)$ since $1/q = 1/p - \alpha/n$. Moreover, if $p > 1$, it is equivalent to condition (1.3) for $(v^{q/p}, v)$ by Theorem (1.7). Hence, Theorem (8.1) follows from Theorems (1.4) and (1.5).

The sufficiency part of the principal result in [**12**] can be obtained from Theorem (1.11). The result states that if $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, then the condition

$$\left(\frac{1}{|I|} \int_I v^{q/p}\, dx \right)^{1/q} \left(\frac{1}{|I|} \int_I v^{-p'/p}\, dx \right)^{1/p'} \leqslant c, \tag{8.2}$$

$1/p + 1/p' = 1$, is necessary and sufficient for $\|I_\alpha f\|_{L^q_{v^{q/p}}} \leqslant c\|f\|_{L^p_v}$. It is simple to check that condition (8.2) is equivalent to the two conditions $v \in A_p$ and $v \in \mathrm{RH}_{q/p}$. The fact that $v \in \mathrm{RH}_{q/p}$ with $1/q = 1/p - \alpha/n$ implies that the pair $(v^{q/p}, v)$ satisfies condition (1.2), and, therefore, by Theorem (1.7), also satisfies condition (1.3). Thus, the fact that (8.2) is sufficient for $\|I_\alpha f\|_{L^q_{v^{q/p}}} \leqslant c\|f\|_{L^p_v}$ follows from Theorem (1.11).

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